Boundary-value problems of enhanced backscattering in a random medium and the inner structure of the Bethe-Salpeter equation

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(Received 27 September 1993; revised manuscript received 14 November 1994)

Boundary-value problems of enhanced backscattering in a system consisting of a random medium and boundaries are investigated, based on the Bethe-Salpeter (BS) equation formalism. The solutions are shown to be obtained, independent of the boundary-value problems involved, from the incoherent part of the solutions when normal scattering is assumed throughout, and also from the variety of expressions that are available to choose from, according to a previous theory that was developed to write various results in a unified form, so that the medium and boundaries are involved on exactly the same basis. The method is based on the coordinate-interchange principle, and the procedure is simple when the interchange is made through optical expressions. The BS equation of the second-order Green's function can be regarded as a four-coordinate function equation. This observation leads us to write basic equations in a manifestly invariant form against an arbitrary coordinate interchange of the four coordinates involved. A detailed inner structure of the BS equation is found therefrom, independently of the specific medium involved. A close relationship with the fourth-order Green's function is shown.

PACS number(s): 03.20.+i, 05.40.+j, 03.80.+r, 05.60.+w

I. INTRODUCTION

In previous papers [1], scattering by a system of random media with rough boundaries was investigated, wherein a scattering matrix of the entire system was constructed by successive addition of independent scattering matrices of the medium and the boundaries, together with the optical condition of each scattering matrix involved, as well as that of the entire system as one scatterer. The theory was developed based on the Bethe-Salpeter (BS) equation, which was introduced for a system of random layers with a possible fixed scatterer embedded [1,3] and in which the random medium and the rough boundaries were treated on exactly the same footing so that several expressions are possible for the same quantity by interchanging the roles of the medium and the boundaries, providing us with a variety of expressions to choose from. Based on the reciprocity principle, enhanced back-scattering was understood as a natural consequence of requiring the coordinate-interchange invariance for the solution of the BS equation, as had been emphasized by Vollhardt and Wölfe [4] in connection with the Anderson localization problem in condensed matter [5,6]. The cyclic diagrams had been introduced and evaluated previously to account for the enhanced backscattering [7]. Also for light waves, enhanced backscattering by a random medium has been investigated both theoretically and experimentally [1,8–10] and the coordinate-interchange principle was utilized to derive the results in a simple manner, within the diffusion approximation. Here the boundary condition for the diffusion equation changes from one boundary to another, depending on a surface impedance determined by the boundary scattering matrix [1]. Enhanced backscattering by a fixed scatterer embedded in a random medium was investigated in detail [10] and effective scattering matrices of the scatterer for both the normal and the enhanced backscattering were introduced, which include effects of the multiple scattering between the scatterer and the surrounding random medium, as well as the shadowing effect, with particular attention to detailed equations of power conservation involved.

In this paper, basic equations are first briefly reviewed by following the procedure of previous papers [1–3], together with alternative versions of the equations (Sec. II). The BS equation thus derived for the second-order Green's function is naturally a two-coordinate matrix equation and is rewritten as a four-coordinate function equation which enables us to formulate equations of the enhanced backscattering systematically in a unified form for a composite system of random layers and rough boundaries (Sec. III). The BS equation is further rewritten in a perfectly symmetrical form with respect to the four coordinates involved, leading to a detailed inner structure of the BS equation and related optical relation (Sec. V).

II. BASIC EQUATIONS

The coordinate vector in three-dimensional space is denoted by $\hat{\mathbf{x}}=(x_1,x_2,x_3)=(\boldsymbol{\rho},z)$ with $\boldsymbol{\rho}=(x_1,x_2)$ and $z=x_3$, where the z axis is taken in the direction normal to the average boundaries (Fig. 1). The scalar product of two space vectors $\hat{\mathbf{a}}=(\mathbf{a},z_z)$ and $\hat{\mathbf{b}}=(\mathbf{b},b_z)$ is denoted by $\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=\mathbf{a}\cdot\mathbf{b}+a_zb_z$, where $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2$. We first consider two random layers separated by a rough boundary which is planar on average, as illustrated in Fig. 1. A scalar wave function $\psi(\hat{\mathbf{x}})e^{i\omega t}$, where $\omega>0$ and t is time, is considered and is denoted in each layer by $\psi_a(\hat{\mathbf{x}})$, $a=1,2$, whose wave equation is

$$[\mathcal{L}_a-q_a(\hat{\mathbf{x}})]\psi_a(\hat{\mathbf{x}})=j_a(\hat{\mathbf{x}})\ , \qquad (2.1a)$$

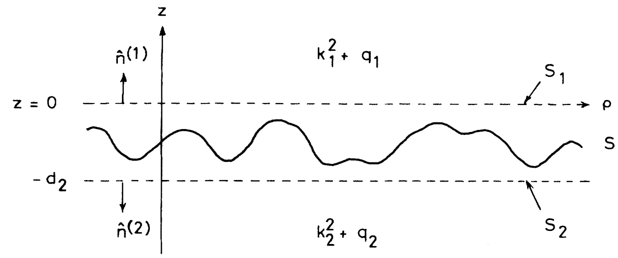

FIG. 1. Geometry of a rough boundary for Eq. (2.4). The real boundary S is distributed within the range $0>z>-d_2$.

$$\mathcal{L}_a=-\left[\frac{\partial}{\partial\hat{\mathbf{x}}}\right]^2-k_a^2,\quad \mathrm{Im}[k_a]<0\ . \tag{2.1b}$$

Here $q_a=q_a^*$ is the random part of the medium and j_a is a source term; k_a is the propagation constant when the medium is free from the random part and the medium is assumed to be nondissipative for the time being. The boundary condition is first assumed to be the continuity of ψ_a and its gradient normal to the (real) boundary surface and, consistently with this, the power flux vector $\hat{\mathbf{W}}_a(\hat{\mathbf{x}})$ in the k_a space is defined by

$$\hat{\mathbf{W}}_a(\hat{\mathbf{x}})=\psi_a^*\hat{\alpha}\psi_a(\hat{\mathbf{x}})\ , \tag{2.2a}$$

with a vector operator $\hat{\alpha}$ defined by

$$\hat{\alpha}=(2i)^{-1}\left[\frac{\overleftarrow{\partial}}{\partial\hat{\mathbf{x}}}-\frac{\overrightarrow{\partial}}{\partial\hat{\mathbf{x}}}\right]\ , \tag{2.2b}$$

where the left and right overarrows mean the operation on the left- and right-hand sides, respectively. Hence the power equation is

$$\frac{\partial}{\partial\hat{\mathbf{x}}}\cdot\sum_a\hat{\mathbf{W}}_a(\hat{\mathbf{x}})=\sum_a(2i)^{-1}[\psi_a^*j_a(\hat{\mathbf{x}})-\psi_aj_a^*(\hat{\mathbf{x}})]\ , \tag{2.3}$$

except the boundary. Here $\hat{\mathbf{W}}_a(\hat{\mathbf{x}})=0$ for $\hat{\mathbf{x}}$ in $k_b\neq k_a$ space.

The boundary condition can be transferred from the real boundary S onto two reference boundary planes, say, S_1 and S_2 at $z=0$ and $-d_2$, respectively, chosen such that the change of the boundary height is ranged between S_1 and S_2 (Fig. 1); hence, with the notation $\partial_n^{(a)}=\hat{\mathbf{n}}^{(a)}\cdot(\partial/\partial\hat{\mathbf{x}})$, where $\hat{\mathbf{n}}^{(a)}$ is the unit vector directed outward normally to S_a, the boundary equation can be written as [2]

$$-\partial_n^{(a)}\psi_a(\boldsymbol{\rho})=\sum_{b=1}^{2}\int d\boldsymbol{\rho}'B_{ab}^{(12)}(\boldsymbol{\rho}|\boldsymbol{\rho}')\psi_b(\boldsymbol{\rho}')\ . \tag{2.4}$$

Here $\psi_a(\boldsymbol{\rho})$ denotes $\psi_a(\hat{\mathbf{x}})$ bounded on S_a and when the boundary is nondissipative,

$$B_{ab}^{(12)\dagger}(\boldsymbol{\rho}|\boldsymbol{\rho}')\equiv B_{ba}^{(12)*}(\boldsymbol{\rho}'|\boldsymbol{\rho})=B_{ab}^{(12)}(\boldsymbol{\rho}|\boldsymbol{\rho}')\ , \tag{2.5}$$

i.e., the matrix defined by the elements $B_{ab}^{(12)}(\boldsymbol{\rho}|\boldsymbol{\rho}')$ is Hermitian with respect to both the coordinates and the subscripts. This means that $B^{(12)}$ is a real symmetrical matrix in view of having symmetrical matrix elements, as can be shown by applying the Green's theorem to the boundary space enclosed by S_1 and S_2 and using Eq. (2.4) for arbitrary two solutions $\psi_a(\hat{\mathbf{x}})$ and $\psi_a'(\hat{\mathbf{x}})$, say, with the vanishing contour surface integral over both sides of S. Hereafter, the boundary space will be neglected, on letting $d_2\to0$, unless otherwise noted; so that $S_{12}=S_1+S_2$ at $z=0$ represents the two reference boundary planes together. The wave equations (2.1) and the boundary equation (2.4) can be written by one wave equation of the form [1,3]

$$(\mathcal{L}_a-q_a)\psi_a-\sum_{b=1}^{2}B_{ab}^{(12)}\psi_b=j_a\ . \tag{2.6}$$

Here both $B_{ab}^{(12)}$ and q_a are regarded as $\hat{\mathbf{x}}$-coordinate matrices, defined by the elements

$$B_{ab}^{(12)}(\hat{\mathbf{x}}|\hat{\mathbf{x}}')=\delta(z+d_a)B_{ab}^{(12)}(\boldsymbol{\rho}|\boldsymbol{\rho}')\delta(z'+d_b)\ ,d_1=0 \tag{2.7}$$

and $q_a(\hat{\mathbf{x}}|\hat{\mathbf{x}}')=q_a(\hat{\mathbf{x}})\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}')$. The solution is subject to the boundary condition that $\partial_n^{(a)}\psi_a=0$, $a=1,2$, inside the boundary space $0>z>-d_2$. The proof can be given by integrating Eq. (2.6) with respect to z over two infinitesimal regions enclosing S_1 and S_2, separately; hence Eq. (2.4) is reproduced.

With a matrix v defined by the elements

$$v_{ab}=q_a\delta_{ab}+B_{ab}^{(12)}\ , \tag{2.8}$$

the equation of the deterministic Green's function for the wave equation (2.6), $g_{ab}(\hat{\mathbf{x}}|\hat{\mathbf{x}}')$, can be written as

$$\sum_c(\mathcal{L}_a\delta_{ac}-v_{ac})g_{cb}(\hat{\mathbf{x}}|\hat{\mathbf{x}}')=\delta_{ab}\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}')\ , \tag{2.9a}$$

or in matrix form as

$$(\mathcal{L}-v)g=1,\quad v=q+B^{(12)}\ . \tag{2.9b}$$

Here v may be regarded as an effective medium representing both the medium and the boundary on an equal basis. Since v is a symmetrical matrix with respect to both the coordinates and the subscripts, $v^T=v$, the unified wave equation (2.9b) shows that the Green's function is also symmetrical, i.e.,

$$g^T=g,\quad v^T=v\ , \tag{2.10}$$

being subject to the reciprocity.

For a general class of scalar waves, the continuity conditions on the (real) boundary can be reduced to those of ψ_a and $\eta_a^{-1}\partial_n\psi_a$, with some constant η_a depending on the ath medium, and the equations can be similarly formulated without changing the basic form [2].

A. Statistical Green's functions

Equation (2.9b) enables us to obtain the statistical Green's functions in exactly the same form as those in an inhomogeneously random medium v and the results are summarized as follows [1–3]. The averaged version of Eq. (2.9b) can be written as

$$(\mathcal{L}-M)G=1,\quad G=\langle g\rangle \tag{2.11}$$

in terms of an effective medium M of v, defined by

$$MG=\langle vg\rangle,\quad M=M^{(q)}+M^{(12)}\ . \tag{2.12}$$

Here $M^{(q)}$ and $M^{(12)}$ are also defined in the same fashion, by

$$M^{(q)}G=\langle qg\rangle,\quad M^{(12)}G=\langle B^{(12)}g\rangle\ , \tag{2.13}$$

and are approximately equal to the independent contributions from the medium and the boundary, respectively, with the elements $M_a^{(q)}\delta_{ab}$ and $M_{ab}^{(12)}$. More precisely, $M^{(q)}$ includes its change caused by the presence of the boundary $B^{(12)}$, say, $M^{(q,12)}$, which works as an effective change of $B^{(12)}$ due to the medium fluctuation [see also (3.47)]. The situation is the same also for $M^{(12)}$. The matrix M has symmetrical elements as the whole, i.e.,

$$M^T=M,\quad G^T=G\ , \tag{2.14}$$

although $M^{(q)T}\neq M^{(q)}$ and $M^{(12)T}\neq M^{(12)}$, strictly speaking. Dividing $B^{(12)}$ into two parts, $\langle B^{(12)}\rangle+b$, with the deterministic part $\langle B^{(12)}\rangle$ and the random part b, the diagrams of $M^{(q)}$ and $M^{(b)}=M^{(12)}-\langle B^{(12)}\rangle$ are shown in Fig. 2(a) in series (to the approximation of Gaussian statistics).

For the statistical Green's function of second order, defined by

$$I_{ab;cd}(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2')=\langle g_{ac}^*(\hat{\mathbf{x}}_1|\hat{\mathbf{x}}_1')g_{bd}(\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_2')\rangle\ , \tag{2.15a}$$

or in matrix form by

$$I(1;2)=\langle g^*(1)g(2)\rangle \tag{2.15b}$$

(here and hereafter, the subscript 1 is attached to the coordinates of quantities of the complex-conjugate wave function and the subscript 2 is attached to those of the original wave function), we first introduce a matrix Δv, defined by

$$\Delta v=v-M=\Delta q+\Delta B^{(12)}\ , \tag{2.16a}$$

where

$$\Delta q=q-M^{(q)},\quad \Delta B^{(12)}=B^{(12)}-M^{(12)} \tag{2.16b}$$

and employ the expression

$$g=G(1+\Delta vg),\quad \langle \Delta vg\rangle=0 \tag{2.17}$$

for both $g^*(1)$ and $g(2)$ on the right-hand side of Eq. (2.15b). Hence we obtain an expression

$$I(1;2)=G^*(1)G(2)[1+K(1;2)I(1;2)] \tag{2.18}$$

of the form of the Bethe-Salpeter equation, with a matrix $K(1;2)$, defined by

$$K(1;2)I(1;2)=\langle \Delta v^*(1)\Delta v(2)g^*(1)g(2)\rangle\ , \tag{2.19}$$

in the same fashion as M has been defined by Eq. (2.12).

Here the matrix K can be divided into four parts as

$$K=K^{(q)}+K^{(12)}+K^{(q,12)}+K^{(12,q)}\ , \tag{2.20}$$

which are respectively defined with Eq. (2.16a), according to

$$K^{(q)}(1;2)I(1;2)=\langle \Delta q^*(1)\Delta q(2)g^*(1)g(2)\rangle\ , \tag{2.21a}$$

$$K^{(12)}(1;2)I(1;2)=\langle \Delta B^{(12)*}(1)\Delta B^{(12)}(2)g^*(1)g(2)\rangle\ , \tag{2.21b}$$

$$K^{(q,12)}(1;2)I(1;2)=\langle \Delta q^*(1)\Delta B^{(12)}(2)g^*(1)g(2)\rangle\ , \tag{2.21c}$$

$$K^{(12,g)}(1;2)I(1;2)=\langle \Delta B^{(12)*}(1)\Delta q(2)g^*(1)g(2)\rangle\ . \tag{2.21d}$$

To the first-order approximation

$$K^{(q)}(1;2)=\langle q(1)q(2)\rangle\ ,\qquad K^{(12)}(1;2)=\langle b(1)b(2)\rangle\ , \tag{2.22}$$

while $K^{(q,12)}$ and $K^{(12,q)}$ are of higher order. Their dia-

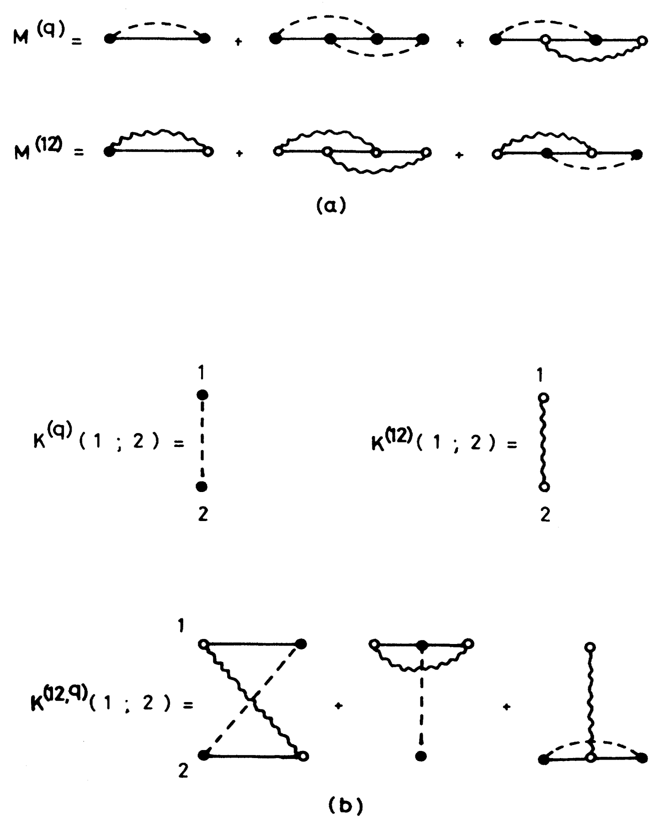

FIG. 2. (a) Schematic diagrams of $M^{(q)}$ and $M^{(b)}=M^{(12)}-\langle B^{(12)}\rangle$, defined by (2.13), are shown to the fourth order of q and b, assuming Gaussian statistics. Here G, q, and b are represented, respectively, by a solid line, filled circles, and open circles and are connected in the order of their matrix multiplication. $\langle q\cdots q\rangle$ is represented by dashed lines connecting the filled circles of the q's and $\langle b\cdots b\rangle$ is represented by wavy lines connecting the open circles of the b's. The terms from $M^{(q,12)}$ and $M^{(12,q)}$ are included by the last diagram of each, respectively. (b) Nonvanishing elements of $K^{(q)}(1;2)$, $K^{(12)}(1;2)$, and $K^{(12,q)}(1;2)$ defined by (2.21) are shown to the lowest order of approximation, with the same notation as in (a).

grams are shown in Fig. 2(b) to the same approximation as in Fig. 2(a) for $M^{(q)}$ and $M^{(b)}$. They play an essential role in the enhanced backscattering and we will return to this problem in Sec. III. For the time being, however, we approximate $K(1;2)$ by an independent sum of $K^{(q)}$ from the medium and $K^{(12)}$ from the boundary, as in (2.31).

The matrices M and K, as defined by Eqs. (2.12) and (2.19), respectively, are not quite independent of each other, subject to a local (optical) relation of the form

$$\frac{\partial}{\partial \rho}\cdot\boldsymbol{\beta}(\hat{\mathbf{x}})=\Gamma(\hat{\mathbf{x}})-\Delta G(\hat{\mathbf{x}})K \ . \tag{2.23}$$

Here the matrices $\Delta G(\hat{\mathbf{x}}|1;2;)$ and $\Gamma(\hat{\mathbf{x}}|1;2)$ are defined by

$$\Delta G(\hat{\mathbf{x}}|1;2)=\delta(\hat{\mathbf{x}}|1;2)(2i)^{-1}[G^*(1)-G(2)] \ , \tag{2.24}$$

$$\Gamma(\hat{\mathbf{x}}|1;2)=\delta(\hat{\mathbf{x}}|1;2)(2i)^{-1}[M^*(1)-M(2)] \ , \tag{2.25}$$

wherein $\delta(\hat{\mathbf{x}}|1;2)$ is defined by the elements

$$\delta_{ab}(\hat{\mathbf{x}}|\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2)=\delta_{ab}\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}_1)\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}_2) \tag{2.26}$$

such that, for any matrices $A^*(1)$ and $B(2)$, the product

$$\delta(\hat{\mathbf{x}}|1;2)A^*(1)B(2)\equiv A^*B(\hat{\mathbf{x}}|1;2)\equiv A^*B(\hat{\mathbf{x}}) \tag{2.27a}$$

represents

$$\sum_{a,b}\int d\hat{\mathbf{x}}_1 d\hat{\mathbf{x}}_2\delta_{ab}(\hat{\mathbf{x}}|\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2)A^*_{ac}(\hat{\mathbf{x}}_1|\hat{\mathbf{x}}'_1)B_{bd}(\hat{\mathbf{x}}_2|\hat{\mathbf{x}}'_2)$$
$$=\sum_a A^*_{ac}(\hat{\mathbf{x}}|\hat{\mathbf{x}}'_1)B_{ad}(\hat{\mathbf{x}}|\hat{\mathbf{x}}'_2) \ . \tag{2.27b}$$

Hence, with a matrix $\hat{\boldsymbol{\alpha}}(\hat{\mathbf{x}})$ defined by the elements

$$\hat{\alpha}_{ab}(\hat{\mathbf{x}}|\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2)=\delta_{ab}(\hat{\mathbf{x}}|\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2)(2i)^{-1}\left[\frac{\partial}{\partial\hat{\mathbf{x}}_1}-\frac{\partial}{\partial\hat{\mathbf{x}}_2}\right] , \tag{2.28}$$

the BS equation (2.18) leads to

$$\frac{\partial}{\partial\hat{\mathbf{x}}}\cdot(\hat{\boldsymbol{\alpha}}+\boldsymbol{\beta})I(\hat{\mathbf{x}})=\Delta G(\hat{\mathbf{x}}) \ , \tag{2.29}$$

equivalent to the averaged version of power equation (2.3), except the β term, which represents an additional power flux by a surface wave propagating along the boundary.

A local optical relation similar to (2.23) holds true also for each $M^{(\,)}$ and $K^{(\,)}$, approximately, providing the optical condition of each constituent; e.g., for $M^{(q)}$ and $K^{(q)}$, Eq. (2.23) is replaced by

$$\Gamma^{(q)}(\hat{\mathbf{x}})-\Delta G^{(0)}(\hat{\mathbf{x}})K^{(q)}=0 \ . \tag{2.30}$$

Here $G^{(0)}$ is the Green's function in a homogeneous space of $M^{(q)}$ and the replacement of $M\rightarrow M^{(q)}$ and $K\rightarrow K^{(q)}$ has been made in view of the negligible boundary effect on $M^{(q)}$ and $K^{(q)}$. The local relation (2.30) leads to the conventional optical relation of the medium cross section per unit volume, by the $\hat{\mathbf{x}}$ integration and subsequent optical transformation (Appendix B). The same also holds for the boundary counterpart.

B. Case of neglecting the enhanced backscattering

The terms $K^{(q,12)}$ and $K^{(12,q)}$ in (2.20) for the incoherent factor K are of higher order and negligible, as long as the enhanced backscattering is not considered. Hereafter in this section, we approximate $K(1;2)$ by an independent sum of $K^{(q)}$ and $K^{(12)}$ as

$$K(1;2)\simeq K^{(q)}(1;2)+K^{(12)}(1;2) \ . \tag{2.31}$$

Here $K^{(q)}$ is a diagonal matrix with respect to the subscripts, having only the elements $K_a^{(q)}\equiv K_{aa;aa}^{(q)}$, while the important elements of $K^{(12)}$ are $K_{ab}^{(12)}\equiv K_{aa;bb}^{(12)}$. Hence, in terms of the notation $I_{ab}^{(q+12)}=I_{aa;bb}$ and

$$U_{ab}^{(C)}(1;2)=G_{ab}^*(1)G_{ab}(2) \ , \tag{2.32}$$

the BS equation (2.18) can be written in 2×2 matrix form as

$$I^{(q+12)}=U^{(C)}[1+(K^{(q)}+K^{(12)})I^{(q+12)}] \ . \tag{2.33}$$

The situation is the same also for the case of a random layer, as illustrated in Fig. 3, and various equations formally remain unchanged with setting

$$M=M^{(q)}+M^{(12)}+M^{(23)} \ , \tag{2.34}$$

$$K\simeq K^{(q)}+K^{(12)}+K^{(23)} \ . \tag{2.35}$$

Thus, using the notation $I_{ab}^{(q+12+23)}$, $a,b=1,2,3$, for the second-order Green's function in this case, we obtain the BS equation in 3×3 matrix form as

$$I^{(q+12+23)}=U^{(C)}[1+(K^{(q)}+K^{(12)}+K^{(23)})I^{(q+12+23)}] \ . \tag{2.36}$$

Here $K^{(q)}$ is a diagonal matrix with the elements $K_{ab}^{(q)}=K_a^{(q)}\delta_{ab}$ and $K^{(12)}$ and $K^{(23)}$ are the contributions purely from the boundaries S_{12} and S_{23}, with the nonvanishing elements $K_{ab}^{(12)}$, $a,b=1,2$, and $K_{ab}^{(23)}$, $a,b=2,3$.

C. Solutions and scattering matrices

To obtain the solution of the BS equation (2.33), we first introduce the solution in the special case $K^{(q)}=0$ (on keeping $M^{(q)}\neq0$), say, $I^{(12)}$, so that

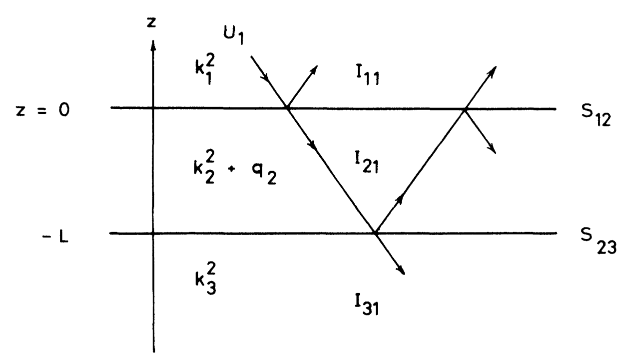

FIG. 3. Geometry and notations of a random layer for Eqs. (2.36), (2.74), and (2.75).

$$I^{(12)}=U^{(C)}(1+K^{(12)}I^{(12)}) , \quad (2.37)$$

with the solution

$$I^{(12)}=U^{(C)}+U^{(C)}+U^{(C)}S^{(12)}U^{(C)} , \quad (2.38)$$

in terms of an (incoherent) scattering matrix $S^{(12)}$ of $K^{(12)}$, defined by

$$S^{(12)}=K^{(12)}(1+U^{(C)}S^{(12)}) \quad (2.39)$$

and given formally by

$$S^{(12)}=(1-K^{(12)}U^{(C)})^{-1}K^{(12)} . \quad (2.40)$$

The Green's function G_{ab} can be written in the same form

$$G_{ab}=G_a^{(0)}\delta_{ab}+G_a^{(0)}T_{ab}^{(12)}G_b^{(0)} , \quad (2.41)$$

in terms of Green's function $G_a^{(0)}$in a homogeneous medium of $M_a^{(q)}$ and a boundary scattering matrix $T_{ab}^{(12)}$ (Appendix A is devoted to deriving a specific expression of $T^{(12)}$ in terms of $M^{(12)}$). Therefore, by introducing a diagonal matrix $U_a(1;2)$, defined by the elements

$$U_a(1;2)=[G_a^{(0)}(1)]^*G_a^{(0)}(2) , \quad (2.42)$$

$U^{(C)}$ of (2.32) can also be written in the form

$$U^{(C)}(1;2)=U(1;2)+U(1;2)V^{(12)}(1;2)U(1;2) , \quad (2.43)$$

with a coherent scattering matrix $V^{(12)}$ defined by

$$V^{(12)}(1;2)=T^{(12)*}(1)T^{(12)}(2)+T^{(12)*}(1)[G^{(0)}(2)]^{-1} + T^{(12)}(2)[G^{(0)*}(1)]^{-1} . \quad (2.44)$$

Herein the interference terms are negligible when the source and the observer are both separated enough from the boundary, while they are otherwise not negligible [e.g., (2.46) and, in the case of shadowing, (3.55)].

Thus, with (2.43), Eq. (2.38) can be written in the form

$$I^{(12)}=U+U\sigma^{(12)}U . \quad (2.45)$$

Here $\sigma^{(12)}$ means a resultant scattering matrix of the boundary S_{12} and is given by

$$\sigma^{(12)}=V^{(12)}+(1+V^{(12)}U)S^{(12)}(UV^{(12)}+1) . \quad (2.46)$$

The introduction of $I^{(12)}$ by Eq. (2.37) enables the BS equation (2.33) to be rewritten as

$$I^{(q+12)}=I^{(12)}(1+K^{(q)}I^{(q+12)}) \quad (2.47)$$

and hence the solution as

$$I^{(q+12)}=I^{(12)}+I^{(12)}S^{(q/12)}I^{(12)} , \quad (2.48)$$

in terms of a scattering matrix $S^{(q/12)}$ of $K^{(q)}$, defined by

$$K^{(q)}I^{(q+12)}=S^{(q/12)}I^{(12)} \quad (2.49)$$

and hence governed by

$$S^{(q/12)}=K^{(q)}(1+I^{(12)}S^{(q/12)}) , \quad (2.50)$$

with the superscript $(q/12)$ meaning the dependence on $\sigma^{(12)}$ through $I^{(12)}$. Here the effect of $\sigma^{(12)}$ can be made explicit by introducing a solution of Eq. (2.50) in the case $\sigma^{(12)}=0$, say, $S^{(0,q)}$, defined by

$$S^{(0q)}=K^{(q)}(1+US^{(0q)})=(1-K^{(q)}U)^{-1}K^{(q)} , \quad (2.51)$$

so that Eq. (2.50) is written, on using (2.45), as

$$S^{(q/12)}=S^{(0q)}(1+U\sigma^{(12)}US^{(q/12)}) \quad (2.52a)$$

$$=(1-S^{(0q)}U\sigma^{(12)}U)^{-1}S^{(0q)} . \quad (2.52b)$$

Thus $I^{(q+12)}$ of (2.48) is finally written, with Eq. (2.45) for $I^{(12)}$, in the form

$$I^{(q+12)}=I^{(12)}+(1+U\sigma^{(12)})\mathcal{J}^{(q/12)}(\sigma^{(12)}U+1) . \quad (2.53)$$

Here the entire effect of the random medium appears only through a new matrix $\mathcal{J}^{(q/12)}$, defined by

$$\mathcal{J}^{(q/12)}=US^{(q/12)}U \quad (2.54)$$

and given as the solution of

$$\mathcal{J}^{(q/12)}=\mathcal{J}^{(0q)}(1+\sigma^{(12)}\mathcal{J}^{(q/12)}) , \quad (2.55)$$

where, from Eq. (2.51), $\mathcal{J}^{(0q)}$ is the solution of

$$\mathcal{J}^{(0q)}=US^{(0q)}U \quad (2.56a)$$

$$=UK^{(q)}(U+\mathcal{J}^{(0q)}) \quad (2.56b)$$

and is a diagonal matrix with respect to the subscripts, each matrix element of which is the independent solution in a semi-infinite random layer of q_1 $(z>0)$ or q_2 $(z<0)$, and tends to zero as $K^{(q)}\rightarrow 0$.

1. Example: Case of a semi-infinite random layer ($q_1=0$, $q_2\neq 0$)

We have $\mathcal{J}_1^{(0q)}=\mathcal{J}_{1b}^{(q/12)}=0$, $b=1,2$, and the only nonvanishing matrix element of $\mathcal{J}^{(q/12)}$ is $\mathcal{J}_{22}^{(q/12)}$, which, from Eq. (2.55), is the solution of the integral equation

$$\mathcal{J}_{22}^{(q/12)}=\mathcal{J}_2^{(0q)}(1+\sigma_{22}^{(12)}\mathcal{J}_{22}^{(q/12)}) . \quad (2.57)$$

Here $\mathcal{J}_2^{(0q)}$ is the solution of Eq. (2.56b); hence

$$\mathcal{J}_2^{(0q)}=U_2K_2^{(q)}(U_2+\mathcal{J}_2^{(0q)}) , \quad (2.58)$$

which, to the optical approximation, can be converted to the radiative transfer equation with an incoherent source term and subject to the condition of no reflection at the boundary of the medium $K^{(q)}$, distributed over the range $0\geq z\geq -\infty$ (Fig. 1). Hence, when the wave source is located in k_1 space, $I^{(q+12)}$ in the same space is given according to (2.53), by

$$I_{11}^{(q+12)}=I_{11}^{(12)}+U_1\sigma_{12}^{(12)}\mathcal{J}_{22}^{(q/12)}\sigma_{21}^{(12)}U_1 \quad (2.59)$$

and the wave transmitted into the k_2 space is

$$I_{21}^{(q+12)}=I_{21}^{(12)}+(1+U_2\sigma_{22}^{(12)})\mathcal{J}_{22}^{(q/12)}\sigma_{21}^{(12)}U_1 . \quad (2.60)$$

Another expression of $I^{(q+12)}$ is obtained by interchanging the roles of the medium q and the boundary S_{12} and is given by

$$I^{(q+12)}=I^{(0q)}+I^{(0q)}\sigma^{(12/q)}I^{(0q)} . \quad (2.61)$$

Here $I_{ab}^{(0q)}=I_a^{(0q)}\delta_{ab}$, $I_a^{(0q)}=U_a+\mathcal{J}_a^{(0q)}$, with the incoherent part $\mathcal{J}_a^{(0q)}$ of Eqs. (2.56), is the solution of the BS equation in a homogeneous medium of $K_a^{(q)}$ when $\sigma^{(12)}=0$ and the factor $\sigma^{(12/q)}$ means an effective scattering matrix of $\sigma^{(12)}$ as affected by the medium fluctuation; it is defined by

$$\sigma^{(12)}\mathcal{J}^{(q/12)}=\sigma^{(12/q)}\mathcal{J}^{(0q)} , \qquad (2.62a)$$

$$\mathcal{J}^{(q/12)}\sigma^{(12)}=\mathcal{J}^{(0q)}\sigma^{(12/q)} , \qquad (2.62b)$$

which enables Eq. (2.55) to be written by

$$\mathcal{J}^{(q/12)}=\mathcal{J}^{(0q)}+\mathcal{J}^{(0q)}\sigma^{(12/q)}\mathcal{J}^{(0q)} , \qquad (2.63)$$

together with the inverse relation

$$\sigma^{(12/q)}=\sigma^{(12)}+\sigma^{(12)}\mathcal{J}^{(q/12)}\sigma^{(12)} . \qquad (2.64)$$

Substitution of relation (2.63) into Eq. (2.62a) leads to

$$\sigma^{(12/g)}=\sigma^{(12)}[1+\mathcal{J}^{(0q)}\sigma^{(12/q)}] , \qquad (2.65)$$

which formally gives $\sigma^{(12/q)}$, by

$$\sigma^{(12/q)}=[1-\sigma^{(12)}\mathcal{J}^{(0q)}]^{-1}\sigma^{(12)} , \qquad (2.66)$$

in terms of $\sigma^{(12)}$ and the boundary values of $\mathcal{J}^{(0q)}$ on S_{12}. When $q_1=0$ and $q_2\neq 0$, Eq. (2.61) shows that

$$I_{11}^{(q+12)}=U_1+U_1\sigma_{11}^{(12/q)}U_1 , \qquad (2.67)$$

which provides an alternative expression of (2.59).

Also for the case of three random layers, as illustrated in Fig. 3, the situation becomes the same by introducing a solution of when $K_a^{(q)}=0$, $a=1,2,3$, say $I^{(12+23)}$, and letting $I^{(12+23)}$ do all the roles of $I^{(12)}$ in the equations of $I^{(q+12)}$; that is, the basic equations (2.53)–(2.56) remain unchanged with the replacement of the superscript (12) by (12+23) and using the expression

$$I_{ab}^{(12+23)}=U_a\delta_{ab}+U_a\sigma_{ab}^{(12+23)}U_b . \qquad (2.68)$$

Here, when the distance between the two boundaries L is sufficiently large compared with the wave coherence distance, say, γ_2^{-1}, so that $\gamma_2 L\gg 1$, $\sigma^{(12+23)}$ can be approximated by

$$\sigma^{(12+23)}\simeq\sigma^{(12)}+\sigma^{(23)} , \qquad (2.69)$$

being the independent sum of the two boundary scattering matrices $\sigma^{(12)}$ of S_{12} and $\sigma^{(23)}$ of S_{23}. Thus the solution $I^{(q+12+23)}$ of Eq. (2.36) can be expressed, in reference to Eq. (2.53) for $I^{(q+12)}$, by a 3×3 matrix equation as

$$I^{(q+12+23)}=I^{(12+23)}+(1+U\sigma^{(12+23)})\times\mathcal{J}^{(q/12+23)}(\sigma^{(12+23)}U+1) . \qquad (2.70)$$

Here, with the approximation (2.69), Eq. (2.55) is changed to

$$\mathcal{J}^{(q/12+23)}=\mathcal{J}^{(0q)}[1+(\sigma^{(12)}+\sigma^{(23)})\mathcal{J}^{(q/12+23)}]$$

$$=\mathcal{J}^{(q/12)}[1+\sigma^{(23)}\mathcal{J}^{(q/12+23)}] \qquad (2.71)$$

$$=\mathcal{J}^{(q/12)}+\mathcal{J}^{(q/12)}\sigma^{(23/q+12)}\mathcal{J}^{(q/12)} , \qquad (2.72)$$

with a scattering matrix $\sigma^{(23/q+12)}$ of $\sigma^{(23)}$, defined by

$$\sigma^{(23)}\mathcal{J}^{(q/12+23)}=\sigma^{(23/q+12)}\mathcal{J}^{(q/12)} . \qquad (2.73a)$$

Here it can be shown that

$$\sigma^{(23/q+12)}=[1-\sigma^{(23/g)}\mathcal{J}^{(0q)}\sigma^{(12/q)}\mathcal{J}^{(0q)}]^{-1}\sigma^{(23/q)} , \qquad (2.73b)$$

which has been given exclusively in terms of the effective scattering matrices of the boundaries $\sigma^{(23/q)}$ and $\sigma^{(12/q)}$ and the boundary values of $\mathcal{J}^{(0q)}$ on S_{12} and S_{23}.

2. *Example: Case of a random layer* ($q_1=q_3=0$, $q_2\neq 0$)

The only nonvanishing element of $\mathcal{J}^{(q/12+23)}$ in (2.70) is $\mathcal{J}_{22}^{(q/12+23)}$ in this case. Hence, when the source is in k_1 space and the layer width L is large enough so that $\gamma_2 L\gg 1$, $I^{(q+12+23)}$ within the same space is given, with $I_{11}^{(12+23)}\approx I_{11}^{(12)}$, by

$$I_{11}^{(q+12+23)}=I_{11}^{(12)}+U_1\sigma_{12}^{(12)}\mathcal{J}_{22}^{(q/12+23)}\sigma_{21}^{(12)}U_1 \qquad (2.74a)$$

$$=I_{11}^{(q+12)}+U_1\sigma_{12}^{(12/q)}\mathcal{J}_2^{(0q)}\sigma_{22}^{(23/q+12)}\times\mathcal{J}_2^{(0q)}\sigma_{21}^{(12/g)}U_1 , \qquad (2.74b)$$

where the second expression has been derived by substitution of (2.72) and followed use of relations (2.62) with expression (2.59) for $I_{11}^{(q+12)}$. Thus the second term of Eq. (2.74b) means the entire effect of the boundary S_{23}, in terms of the effective scattering matrix $\sigma^{(12/q)}$ of S_{12} and the corresponding matrix $\sigma^{(23/q+12)}$ of S_{23}, which includes all the effects of multiple scattering between $\sigma^{(23)}$, $\sigma^{(12)}$, and the medium q_2. Similarly, the wave transmitted into k_3 space is given by

$$I_{31}^{(q+12+23)}=U_3\sigma_{32}^{(23)}\mathcal{J}_{22}^{(q/12+23)}\sigma_{21}^{(12)}U_1 \qquad (2.75a)$$

$$=U_3\sigma_{32}^{(23/q+12)}\mathcal{J}_2^{(0q)}\sigma_{21}^{(12/q)}U_1 , \qquad (2.75b)$$

where the contribution from $I_{31}^{(12+23)}$ has been neglected and the second expression is a direct consequence of relation (2.73a).

In Eqs. (2.74a) and (2.75a), the random medium is involved only through the boundary values of $\mathcal{J}_{22}^{(q/12+23)}$ on S_{12} and S_{23} and, in view of the present condition $\gamma_2 L\gg 1$, the latter can be approximately obtained from the boundary-value solution of a diffusion equation subject to appropriate boundary conditions that are determined by $\sigma^{(12)}$ and $\sigma^{(23)}$, as was previously done for the present case in some detail [1]. The same is also true for $\mathcal{J}^{(0q)}$ to numerically evaluate $\sigma^{(12/q)}$, $\sigma^{(23/q)}$ and $\sigma^{(23/q+12)}$, according to Eqs. (2.66) and (2.73b), and thereby to obtain $I_{11}^{(q+12)}$ and $I_{11}^{(q+12+23)}$ according to Eqs. (2.67) and (2.74b), respectively.

III. COORDINATE INTERCHANGE INVARIANCE AND THE ENHANCED BACKSCATTERING

The deterministic Green's function is subject to the reciprocity $g(\hat{\mathbf{x}}|\hat{\mathbf{x}}')=g(\hat{\mathbf{x}}'|\hat{\mathbf{x}})$ or $g^T=g$, in view of $v^T=v$ in the governing equation (2.9b), and therefore not only the first-order Green's function subject to $G(\hat{\mathbf{x}}|\hat{\mathbf{x}}')=G(\hat{\mathbf{x}}'|\hat{\mathbf{x}})$, but also the second-order Green's function $I(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2')$ should be invariant for each of the interchanges $x_1\leftrightarrow x_1'$ and $x_2\leftrightarrow x_2'$, independently. This results in that, based on this simple symmetry alone, we can find a fundamental structure of the basic matrix K to a considerable extent without knowing the details of the specific medium involved.

We first consider the case of a homogeneously random medium q so that $K=K^{(q)}$ [whenever necessary, we can replace the K by that of a composite system, e.g., of (2.20)] and introduce a four-coordinate function $\check{I}(\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_3,\hat{\mathbf{x}}_4)$, defined by

$$\check{I}(\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_3,\hat{\mathbf{x}}_4)=I(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_1'\to\hat{\mathbf{x}}_3;\hat{\mathbf{x}}_2'\to\hat{\mathbf{x}}_4) , \tag{3.1}$$

and similarly a four-coordinate function $\check{U}$ defined by

$$\check{U}(\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_3,\hat{\mathbf{x}}_4)=G^*(\hat{\mathbf{x}}_1|\hat{\mathbf{x}}_3)G(\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_4) . \tag{3.2}$$

Also we rewrite the matrix K in the BS equation (2.18) by using the notation K_{12} to make sure that it is a two-coordinate matrix with respect to $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ with the elements $K(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2')$, on using the primed coordinates for the row, so that $K_{12}\check{I}$ represents

$$\int d\hat{\mathbf{x}}_1'd\hat{\mathbf{x}}_2'K(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2')\check{I}(\hat{\mathbf{x}}_1',\hat{\mathbf{x}}_2',\hat{\mathbf{x}}_3,\hat{\mathbf{x}}_4) . \tag{3.3}$$

In the same way, U will be rewritten by U_{12} when using it in the original meaning and the original I will likewise be rewritten by I_{12} whenever confusing. On the other hand, the matrix K_{12} can also be regarded as the four-coordinate function $\check{K}_{12}(x_1,x_2,x_3,x_4)$, defined in the same way as $\check{I}$ had been defined in terms of $I=I_{12}$ by Eq. (3.1).

Thus the BS equation (2.18) can be written by

$$\check{I}=\check{U}+U_{12}K_{12}\check{I} , \tag{3.4}$$

with the solution

$$\check{I}=\check{U}+U_{12}U_{34}\check{S} , \tag{3.5}$$

which represents

$$I=U+USU , \tag{3.6}$$

in terms of the scattering matrix S of K, defined by

$$KI=SU,\quad IK=US , \tag{3.7}$$

and given as the solution of

$$S=K(1+US) \tag{3.8a}$$

$$=K+KIK . \tag{3.8b}$$

Here U_{34} is the matrix when U is regarded as a matrix with respect to $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{x}}_4$, say, the $\hat{\mathbf{x}}_3$-$\hat{\mathbf{x}}_4$ matrix, with the elements

$$U(\hat{\mathbf{x}}_3;\hat{\mathbf{x}}_4|\hat{\mathbf{x}}_3';\hat{\mathbf{x}}_4')=G^*(\hat{\mathbf{x}}_3|\hat{\mathbf{x}}_3')G(\hat{\mathbf{x}}_4|\hat{\mathbf{x}}_4') , \tag{3.9}$$

and therefore commutable with U_{12}, i.e., $U_{12}U_{34}=U_{34}U_{12}$; a function $\check{U}_{34}$ is also defined by (3.9) with $\hat{\mathbf{x}}_3'\to\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_4'\to\hat{\mathbf{x}}_2$, being cyclically changed within the odd and the even numbers, respectively, hence

$$\check{U}=\check{U}_{12}=\check{U}_{34} . \tag{3.10}$$

Hence Eqs. (3.8a) and (3.8b) can be written as

$$\check{S}=\check{K}_{12}+K_{12}U_{12}\check{S} \tag{3.11a}$$

$$=\check{K}_{12}+K_{12}K_{34}\check{I} , \tag{3.11b}$$

in a form similar to Eq. (3.5).

Here we observe that the function $\check{I}$ is invariant against the interchange of $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_4$, as we already noticed based on the reciprocity, and therefore $\check{I}=\check{I}_{12}=\check{I}_{14}$, in view of the matrix elements of I_{14} which are given by those of I_{12} with $\hat{\mathbf{x}}_2\to\hat{\mathbf{x}}_4$ and $\hat{\mathbf{x}}_2'\to\hat{\mathbf{x}}_4'$. Similarly, $\check{U}=\check{U}_{14}$ and

$$U_{12}U_{34}=U_{14}U_{32}=U^{(4)}\equiv G^*(1)G(2)G^*(3)G(4) . \tag{3.12}$$

We can write the conventional reciprocity as $\check{I}_{12}=\check{I}_{34}$, $\check{S}_{12}=\check{S}_{34}$, and $\check{K}_{12}=\check{K}_{34}$, which are the invariance against the simultaneous interchanges $\hat{\mathbf{x}}_1\leftrightarrow\hat{\mathbf{x}}_3$ and $\hat{\mathbf{x}}_2\leftrightarrow\hat{\mathbf{x}}_4$.

Thus we learn from expression (3.5) of $\check{I}$ that

$$\check{S}\equiv\check{S}_{12}=\check{S}_{14} \tag{3.13}$$

and, therefore, by the interchange $\hat{\mathbf{x}}_2\leftrightarrow\hat{\mathbf{x}}_4$ in Eq. (3.11a),

$$\check{S}=\check{K}_{14}+K_{14}U_{14}\check{S} , \tag{3.14}$$

which, upon comparing with the original equation (3.11a), shows that $\check{K}_{14}\neq\check{K}_{12}$ and that $\check{K}_{12}$ can be written in the form [4]

$$\check{K}_{12}=\check{K}^0+K_{14}U_{14}\check{S} , \tag{3.15}$$

with a symmetrical (and irreducible as defined below) matrix K^0 subject to

$$\check{K}^0\equiv\check{K}^0_{12}=\check{K}^0_{14}=\check{K}^0_{34}=\check{K}^0_{32} . \tag{3.16}$$

In fact, the second term of (3.15) is "U_{12} irreducible" in the sense of having no part that can be written in the form $A_2U_{12}B_{12}$, so that its diagram is inseparable into two parts A_{12} and B_{12} by cutting the two lines of $U_{12}=G^*(1)G(2)$ [Fig. 4(a)]. The substitution of Eq. (3.15) into the first term of (3.11a) yields a symmetrical expression of $\check{S}$ [Fig. 4(b)]

$$\check{S}=\check{K}^0+K_{12}U_{12}\check{S}+K_{14}U_{14}\check{S} , \tag{3.17}$$

which, from (3.14), shows that

$$\check{K}_{14}=\check{K}^0+K_{12}U_{12}\check{S} , \tag{3.18}$$

which is the same equation as that obtained from Eq. (3.15) by interchanging $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_4$, as it should be to be consistent. $\check{K}_{14}$ is U_{14} irreducible with the irreducible $\check{K}^0$ with respect to both U_{12} and U_{14}.

Equations (3.7) can be written by the function equations as

$$K_{12}\check{I}=U_{34}\check{S},\quad K_{34}\check{I}=U_{12}\check{S} , \tag{3.19}$$

leading to the relations

$$K_{12}U_{12}\check{S}=K_{34}U_{34}\check{S}=K_{12}K_{34}\check{I} \tag{3.20a}$$

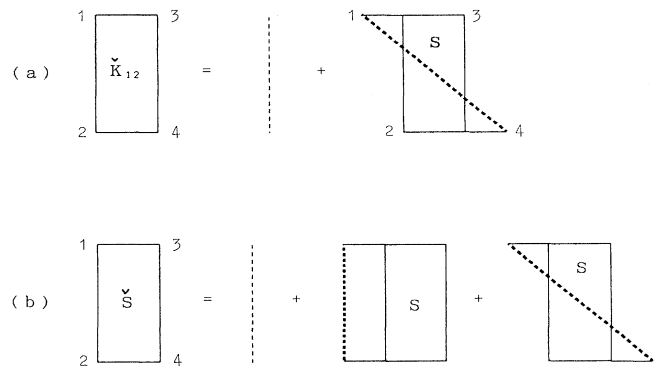

FIG. 4. (a) Diagram of expression (3.15) for $\check{K}_{12}$. The broken lines and the bold broken lines represent $\check{K}^0_{12}$ and $\check{K}_{\dot\nu\mu}$ ($\dot\nu=1,3$; $\mu=2,4$), respectively, and the (horizontal) solid lines represent G^* or G. (b) Diagram of the integral equation (3.17) for $\check{S}$ with the same notations as in (a).

and, by the interchange $\hat{\mathbf{x}}_2 \leftrightarrow \hat{\mathbf{x}}_4$, also

$$K_{14}U_{14}\check{S}=K_{32}U_{32}\check{S}=K_{14}K_{32}\check{I} \ . \tag{3.20b}$$

Thus Eqs. (3.15) and (3.18) can be written by

$$\check{K}_{12}=\check{K}_{34}=\check{K}_0+K_{14}K_{32}\check{I} \ , \tag{3.21a}$$

$$\check{K}_{14}=\check{K}_{32}=\check{K}_0+K_{12}K_{34}\check{I} \tag{3.21b}$$

and Eq. (3.17) by

$$\check{S}=\check{K}_0+K^{(4)}\check{I} \ , \tag{3.22a}$$

in a symmetrical form in terms of a coordinate-interchange-invariant matrix $K^{(4)}$, defined by

$$K^{(4)}=K_{12}K_{34}+K_{14}K_{32} \ . \tag{3.22b}$$

The corresponding equation for $\check{I}$ is given from (3.5) with $U^{(4)}$ of (3.12), by

$$\check{I}=\check{U}+U^{(4)}\check{S} \ . \tag{3.23}$$

Equations (3.22a) and (3.23), which are written in a manifestly coordinate-interchange-invariant form, correspond to the original equations (3.8b) and (3.6), respectively, and provide a basic set of equations to find the four-coordinate functions $\check{S}$ and $\check{I}$, with the approximate $K^{(4)}$ obtained by setting $\check{K}\approx\check{K}^0$; the latter approximation can be avoided with Eqs. (3.21) for $K_{\dot\nu\mu}$ (where $\dot\nu=1,3$ and $\mu=2,4$) at the expense of solving the resulting nonlinear equations of $\check{S}$ and $\check{I}$. Thus we obtain a governing equation for $\check{S}$ in the form

$$\check{S}=\check{K}^{(1)}+K^{(4)}U^{(4)}\check{S} \ , \tag{3.24a}$$

$$\check{K}^{(1)}=\check{K}^0+K^{(4)}\check{U} \ . \tag{3.24b}$$

A. Approximation

In this section we consider a complex system of random medium and boundaries; hence, all the U's in the preceding section are changed to $U^{(C)}$, defined by (2.32) and written in the form (2.43). We also make the approximations $K_{12}\approx K^0_{12}$ and $K_{14}\approx K^0_{14}$ in Eq. (3.17) to regard it as a linear equation of $\check{S}$. Thus

$$\check{S}=\check{K}_{12}+K^0_{12}U^{(C)}_{12}\check{S} \ , \tag{3.25}$$

$$\check{K}_{12}=\check{K}^0+K^0_{14}U^{(C)}_{14}\check{S} \ , \tag{3.26}$$

where Eq. (3.26) is from (3.15).

To find an approximate $\check{K}_{12}$ from (3.26), we substitute expression (3.25) for $\check{S}$, hence

$$\check{K}_{12}\simeq\check{K}^0+K^0_{14}U^{(C)}_{14}\check{K}_{12} \tag{3.27}$$

upon neglecting the term $K^0_{14}U^{(C)}_{14}K^0_{12}U^{(C)}_{12}\check{S}$, whose integrated contribution is generally small and is neglected in consequence of being a cross product of $K^0_{14}U^{(C)}_{14}$ and $K^0_{12}U^{(C)}_{12}\check{S}$. The solution of Eq. (3.27) can be written as

$$\check{K}_{12}=\check{S}^0_{14}\equiv(1-K^0_{14}U^{(C)}_{14})^{-1}\check{K}^0 \tag{3.28a}$$

$$=\check{K}^0+K^0_{14}U^{(C)}_{14}\check{S}^0_{14} \ , \tag{3.28b}$$

in terms of the $\check{S}^0_{14}$ that can be obtained, on making the interchange $\hat{\mathbf{x}}_2 \leftrightarrow \hat{\mathbf{x}}_4$, from the solution $\check{S}^0_{12}$ of Eq. (3.8a) in the case $\check{K}\approx\check{K}^0$.

Summarizing, the result (3.28b) can be written in the form

$$K=K^0+\Delta K \ . \tag{3.29}$$

Here

$$\Delta\check{K}=\check{S}^0_{14}-\check{K}^0=K^0_{14}U^{(C)}_{14}\check{S}^0_{14} \tag{3.30a}$$

$$=K^0_{14}K^0_{32}\check{I}^0 \tag{3.30b}$$

and the substitution of (3.29) into (3.25) leads to

$$S\simeq S^0+\Delta K \ . \tag{3.31}$$

Hence the resulting I from (3.5) can also be divided into two parts as

$$I=I^0+I^{(\mathrm{back})} \ . \tag{3.32}$$

Here I^0 is the term due to the normal scattering and, from Eq. (3.30a),

$$\check{I}^{(\mathrm{back})}=U^{(4,C)}\Delta\check{K} \tag{3.33}$$

$$=\check{\mathcal{I}}^0_{14}-U^{(4,C)}\check{K}^0 \ , \tag{3.34}$$

in which $U^{(4,C)}$ is the same as $U^{(4)}$ of (3.12) except that U has been replaced by $U^{(C)}$ of (2.43); the first term is that which is obtained from $\check{\mathcal{I}}^0$, the incoherent part of $\check{I}^0$, by the interchange $\hat{\mathbf{x}}_2 \leftrightarrow \hat{\mathbf{x}}_4$; the last term is a contribution by the single scattering by K^0, which will often be neglected hereinafter. Thus we learned that, to find the term $I^{(\mathrm{back})}$ which is responsible for the enhanced backscattering, we only need the incoherent term in the normal scattering, independent of the boundary-value problems involved.

B. Case of a semi-infinite random layer ($q_1=0, q_2\neq 0$)

As the basic matrix K^0 in this case, we choose

$$K^0=K^{(q)}+K^{(12)} \tag{3.35}$$

from (2.31); more exactly, the right-hand side should be written as $K^{(0,q)}+K^{(0,12)}$, say, as a sum of independent contributions from the media q and B_{12}. Hence the second-order Green's function in the case of the normal scattering is $I^{(q+12)}=U^{(C)}+\mathcal{J}^{(q+12)}$, with the incoherent part $\mathcal{J}^{(q+12)}$ to be given by (3.38), while the basic matrix K can be written, from Eqs. (3.29) and (3.30), in the form

$$K=K^{(q)}+K^{(12)}+\Delta K\ , \tag{3.36}$$

$$\Delta\check{K}=[(K^{(q)}+K^{(12)})_{14}(K^{(q)}+K^{(12)})_{32}\check{I}^{(q+12)}]\ . \tag{3.37}$$

Here ΔK includes only the second- and higher-order terms of $K^{(q)}+K^{(12)}$.

Shown in Fig. 5(a) are the diagrams of the first several terms of the original (3.37) before the coordinate interchange (assuming Gaussian statistics as in Fig. 2), while shown in Fig. 5(b) are those after the interchange $\hat{\mathbf{x}}_2\leftrightarrow\hat{\mathbf{x}}_4$, which are, therefore, those of the resulting ΔK. In Fig. 5(b) the first term is just the second-order term of $K^{(12)}$ defined by Eq. (2.21b) and the first term of the second line is the first nonvanishing term of $K^{(q,12)}$ defined by Eq. (2.21c); ΔK naturally depends on $V^{(12)}$ to fulfill the boundary conditions involved. Thus $K^{(q,12)}$ and $K^{(12,q)}$ are both produced as a consequence of considering the coordinate-interchange invariance and are nonlocal, long-range functions fulfilling all the boundary conditions involved, in contrast to the basic parts $K^{(q)}$ and $K^{(12)}$ chosen for K^0 in (3.35), which are short-range functions of the order of the medium correlation distances.

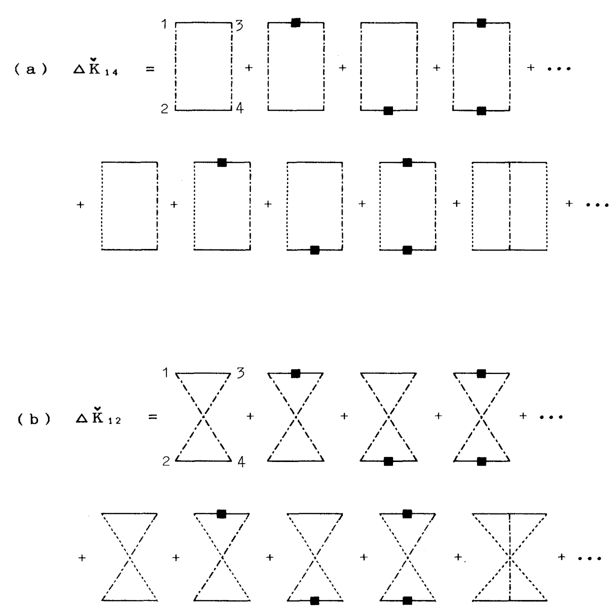

FIG. 5. (a) Diagram series of $\Delta\check{K}_{14}$ from (3.37) before the coordinate interchange $\hat{\mathbf{x}}_2\leftrightarrow\hat{\mathbf{x}}_4$. The dotted lines and the dot-dashed lines represent $K^{(q)}$ and $K^{(12)}$, respectively, and the square box represents $T^{(12)}$ defined in (2.43) with (2.44). The diagrams in the first line are the contributions purely from the boundary and those in the second line are from the close terms made by the medium and the boundary. (b) The diagrams of $\Delta\check{K}=\Delta\check{K}_{12}$ are shown, which are obtained from those of (a) by the interchange $\hat{\mathbf{x}}_2\leftrightarrow\hat{\mathbf{x}}_4$.

In the present case in which $q_1=0$ and $q_2\neq 0$, we choose expression (2.59) for $I^{(q+12)}$ in the region $z\geq 0$ to give the incoherent part $\mathcal{J}_{11}^{(q+12)}$ by

$$\mathcal{J}_{11}^{(q+12)}=\mathcal{J}_{11}^{(12)}+U_1\sigma_{12}^{(12)}\mathcal{J}_{22}^{(q/12)}\sigma_{21}^{(12)}U_1\ , \tag{3.38}$$

where $\mathcal{J}_{11}^{(12)}$ is the contribution purely from $I_{11}^{(12)}$ [the subscripts in (3.38) refer to the space numbers and not to the coordinate numbers as employed in (3.34)]. Thus, once a specific expression of $\mathcal{J}_{11}^{(q+12)}$ is obtained with an appropriate method including the diffusion approximation, for example, $I^{(\text{back})}$ can be obtained therefrom by the coordinate interchange $\hat{\mathbf{x}}_2\leftrightarrow\hat{\mathbf{x}}_4$ according to Eq. (3.34). Appendix B is devoted to the interchange procedure to derive $I^{(\text{back})}$ from the optical expression.

C. Case of a finite random layer ($q_1=q_3=0, q_2\neq 0$)

With the geometry in this case as shown in Fig. 3, we now choose

$$K^0=K^{(q)}+K^{(12)}+K^{(23)}\ . \tag{3.39}$$

hence Eq. (3.36) and (3.37) are replaced simply by

$$K=K^{(q)}+K^{(12)}+K^{(23)}+\Delta K\ , \tag{3.40a}$$

$$\Delta\check{K}=[(K^{(q)}+K^{(12)}+K^{(23)})_{14}\times(K^{(q)}+K^{(12)}+K^{(23)})_{32}\check{I}^{(q+12+23)}]\ , \tag{3.40b}$$

with the replacement of $K^{(12)}\rightarrow K^{(12)}+K^{(23)}$ and $I^{(q+12)}\rightarrow I^{(q+12+23)}$. Here

$$U^{(C)}=U+UV^{(12+23)}U\ , \tag{3.41a}$$

where

$$V^{(12+23)}\simeq V^{(12)}+V^{(23)}\ , \tag{3.41b}$$

when the two boundaries are separated enough to be $\gamma_2 L\gg 1$, like $\sigma^{(12+23)}$ in (2.69). Hence most of the previous equations remain the same with the replacement of the superscript $(12)\rightarrow(12+23)$. Thus the entire incoherent part of $I_{11}^{(q+12+23)}$, $\mathcal{J}_{11}^{(q+12+23)}$, is given, from (2.74a), by

$$\mathcal{J}_{11}^{(q+12+23)}=\mathcal{J}_{11}^{(12)}+U_1\sigma_{12}^{(12)}\mathcal{J}_{22}^{(q/12+23)}\sigma_{21}^{(12)}U_1\ . \tag{3.42}$$

Here $\mathcal{J}^{(q/12+23)}$ is given by (2.72) in terms of $\mathcal{J}^{(q/12)}$ and the $\sigma^{(23/q+12)}$ of (2.73b), which means an effective scattering matrix of the additional boundary S_{23} at $z=-L$, while, when $\gamma_2 L\gg 1$, $\mathcal{J}_{22}^{(q/12+23)}$ can be directly obtained as the boundary-value solution of a diffusion equation [1].

Thus, once a specific expression of the last term is found, e.g., through its optical expression, the wave intensity due to the enhanced backscattering by the boundary S_{23} can be found based on the principle (3.34), by the coordinate interchange according to (B21) with (B25) and (B26).

D. Fixed scatterer embedded in a semi-infinite random layer

Here we consider the case in which a fixed scatterer $q_\alpha(\hat{\mathbf{x}})$ is embedded in the random layer of $q_2(\hat{\mathbf{x}})$ at $\boldsymbol{\rho}_\alpha=(\boldsymbol{\rho}_\alpha, z=-L)$ with the geometry of Fig. 6, so that the original wave equation (2.1a) is changed, with $q\rightarrow q+q_\alpha$ and $\psi\rightarrow\psi^{(\alpha)}$, to

$$(\mathcal{L}-q_\alpha-q)\psi^{(\alpha)}=j \;, \tag{3.43}$$

except on the boundary S_{12}. Correspondingly, the equation of the deterministic Green's function in this case, say $g^{(\alpha)}$, is changed from Eq. (2.9b) to

$$(\mathcal{L}-q_\alpha-v)g^{(\alpha)}=1, \quad v=q+B^{(12)} \;. \tag{3.44}$$

Thus the equation becomes formally the same as in the previous case of three random layer in which $v=q+B^{(12)}+B^{(23)}$ and, consequently, the followed equations also become written by the same equations with the replacement of $\sigma^{(23)}$ by an effective scattering matrix of the scatterer [1,10].

To obtain the first-order Green's function $G^{(\alpha)}=\langle g^{(\alpha)}\rangle$, we observe that the M, as defined by (2.12), is changed to $M+\Delta M_\alpha$ by an amount ΔM_α, according to

$$\langle vg^{(\alpha)}\rangle \simeq \langle qg^{(\alpha)}\rangle=(M+\Delta M_\alpha)G^{(\alpha)} \;; \tag{3.45}$$

i.e., ΔM_α is a change of M caused by the scatterer and has the diagram as shown in the series in Fig. 7, upon neglecting the corresponding effect by the boundary S_{12}. Hence we can write the average of Eq. (3.44), as

$$(\mathcal{L}-M-q'_\alpha)G^{(\alpha)}=1 \;, \tag{3.46a}$$

$$q'_\alpha=q_\alpha+\Delta M_\alpha \;, \tag{3.46b}$$

in terms of the effective scatterer q'_α, and the solution as

$$G^{(\alpha)}=G+GT^M_{\alpha'}G \;, \tag{3.47}$$

in terms of $G=G_M$ in the medium M and the scattering matrix $T^M_{\alpha'}$ of q'_α, defined by

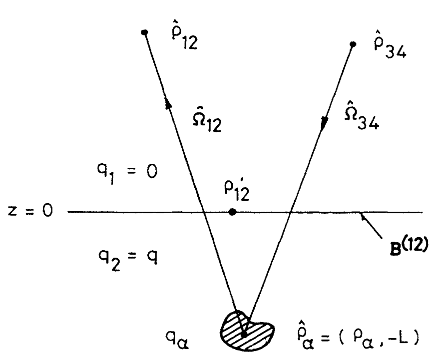

FIG. 6. Geometry of random layer and fixed scatterer for Eq. (3.44).

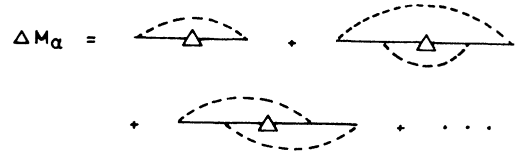

FIG. 7. Diagram of ΔM_α defined by (3.45) for a fixed scatterer. Here G, $K\approx K^0$, and $T^M_{\alpha'}$ are, respectively, represented by solid lines, broken curves, and triangles.

$$T^M_{\alpha'}=q'_\alpha(1+G_M T^M_{\alpha'})=(1-q'_\alpha G_M)^{-1}q'_\alpha \;. \tag{3.48}$$

Thus, by the fixed scatterer, $U^{(C)}(1;2)$ of (2.43) is changed to $U^{(C,\alpha)}(1;2)$, say, of the form

$$U^{(C,\alpha)}=U+UV^{(\alpha+12)}U \;. \tag{3.49a}$$

Here, when $\gamma_2 L\gg 1$, $V^{(\alpha+12)}\approx V^{(\alpha)}+V^{(12)}$; hence we can write

$$U^{(C,\alpha)}=U^{(C)}+\Delta U^{(\alpha)}, \quad \Delta U^{(\alpha)}=UV^{(\alpha)}U \;, \tag{3.49b}$$

where $\Delta U^{(\alpha)}$ means the change caused by the scatterer and $V^{(\alpha)}(1;2)$ is given by the same expression as (2.44) for $V^{(12)}(1;2)$, except $T^{(12)}\rightarrow T^M_{\alpha'}$.

Here, to obtain the incoherent part of wave in the present case, say $\mathcal{I}^{(q+12+\alpha)}_{11}$, we observe that another expression (2.74b) for $I^{(q+12+23)}_{11}$ can be conveniently utilized as it is, with the replacement of the superscript $(23)\rightarrow(\alpha)$. Hence

$$\mathcal{I}^{(q+12+\alpha)}_{11}=\mathcal{I}^{(q+12)}_{11}+U_1\sigma^{(12/q)}_{12}\mathcal{J}^{(0q)}_2 V^{(\alpha/q+12)}_{22} \times\mathcal{J}^{(0q)}_2\sigma^{(12/q)}_{21}U_1 \;. \tag{3.50}$$

Here $V^{(\alpha/q+12)}$ is given, from (2.73b) with $\sigma^{(23)}\rightarrow V^{(\alpha)}$, by

$$V^{(\alpha/q+12)}_{22}=[1-V^{(\alpha/q)}_{22}\mathcal{J}^{(0q)}_2\sigma^{(12/q)}_{22}\mathcal{J}^{(0q)}_2]^{-1}V^{(\alpha/q)}_{22} \;. \tag{3.51}$$

Here, from (2.66),

$$V^{(\alpha/q)}_{22}=[1-V^{(\alpha)}_{22}\mathcal{J}^{(0q)}_2]^{-1}V^{(\alpha)}_{22} \;, \tag{3.52}$$

which means an effective scattering matrix of the scatterer q'_α embedded in a semi-infinite random layer $0\geq z\geq-\infty$, with a free boundary at $z=0$; it includes the entire effect of the multiple scattering between the scatterer and the random medium, and its optical cross section, say $V^{(\alpha/q)}(\hat{\boldsymbol{\Omega}}|\hat{\boldsymbol{\Omega}}')$ for the wave incident from direction $\hat{\boldsymbol{\Omega}}'$ and scattered in direction $\hat{\boldsymbol{\Omega}}$, becomes critically negative in the shadow direction, in view of the original $V^{(\alpha)}(\hat{\boldsymbol{\Omega}}|\hat{\boldsymbol{\Omega}}')$, which is given by [10]

$$V^{(\alpha)}(\hat{\boldsymbol{\Omega}}|\hat{\boldsymbol{\Omega}}')=\sigma^{(\alpha)}(\hat{\boldsymbol{\Omega}}|\hat{\boldsymbol{\Omega}}')-\gamma^{(\alpha)}(\hat{\boldsymbol{\Omega}})\delta^2(\hat{\boldsymbol{\Omega}}-\hat{\boldsymbol{\Omega}}') \;, \tag{3.53}$$

with the conventional cross section $\sigma^{(\alpha)}(\hat{\boldsymbol{\Omega}}|\hat{\boldsymbol{\Omega}}')$ and the total cross section $\gamma^{(\alpha)}(\hat{\boldsymbol{\Omega}})$ of the scatterer. Hence

$$\int d\hat{\boldsymbol{\Omega}}\,V^{(\alpha)}(\hat{\boldsymbol{\Omega}}|\hat{\boldsymbol{\Omega}}')\sim\int d\hat{\boldsymbol{\Omega}}\,V^{(\alpha/q)}(\hat{\boldsymbol{\Omega}}|\hat{\boldsymbol{\Omega}}')\sim 0 \;, \tag{3.54}$$

meaning that the total scattered power should be nearly zero as the whole. This is a consequence of the shadowing effect which results from the interference terms of $V^{(\alpha)}$ similar to those in (2.44) for $V^{(12)}$ (for details, see

Ref. [10]).

The integral (3.54) has a small negative value and the absorbed power thereby is spent to make the enhanced backscattering by the scatterer. Here the latter wave intensity observed in the k_1 space can be obtained from an optical expression of the second term of (3.50), with the coordinate-interchange method as described in Appendix B.

IV. BASIC EQUATIONS IN AN INVARIANT FORM OF THE COORDINATE INTERCHANGE

The previous equations can be rewritten in a more symmetrical form with respect to the coordinates involved. We first assume a homogeneously random medium with the BS equation of the form (2.18); the extension to the case of a composite system of random medium and boundaries can be achieved simply by the replacement (2.20) or, more briefly, (2.31), as we did in Sec. III. We first introduce the notations $h_{\dot{\nu}\mu}$ and $\pi_{\dot{\nu}\mu}$ defined by

$$h_{\dot{\nu}\mu}=K_{\dot{\nu}\mu}U_{\dot{\nu}\mu}\ , \tag{4.1a}$$

$$\pi_{\dot{\nu}\mu}=1-h_{\dot{\nu}\mu}=1-K_{\dot{\nu}\mu}U_{\dot{\nu}\mu}\ , \tag{4.1b}$$

where $\dot{\nu}=1,3$ and $\mu=2,4$ and observe that Eqs. (3.11a) and (3.14) can be unified to be written by

$$\check{S}=\check{K}_{\dot{\nu}\mu}+h_{\dot{\nu}\mu}\check{S} \tag{4.2}$$

or simply

$$\pi_{\dot{\nu}\mu}\check{S}=\check{K}_{\dot{\nu}\mu}\ , \tag{4.3}$$

which says that the operation of $\pi_{\dot{\nu}\mu}$ on $\check{S}$ yields its $U_{\dot{\nu}\mu}$-irreducible part $\check{K}_{\dot{\nu}\mu}$. From Eqs. (3.15) and (3.18), the $\check{K}_{\dot{\nu}\mu}$'s are written in terms of $\check{K}^0$ and $\check{S}$, by

$$\check{K}_{12}=\check{K}^0+h_{14}\check{S},\quad \check{K}_{14}=\check{K}^0+h_{12}\check{S}\ , \tag{4.4}$$

and $\check{S}$ is governed by Eq. (3.17) or

$$\check{S}=\check{K}^0+(h_{12}+h_{14})\check{S}\ . \tag{4.5}$$

Also, from Eqs. (3.20), there exist the following relations between $\check{S}$, $\check{I}$, and $K_{\dot{\nu}\mu}$:

$$h_{12}\check{S}=h_{34}\check{S}=K_{12}K_{34}\check{I}\ , \tag{4.6a}$$

$$h_{14}\check{S}=h_{32}\check{S}=K_{14}K_{32}\check{I}\ . \tag{4.6b}$$

The BS equation (2.18) as a two-coordinate matrix equation can be generalized to be written by

$$I_{\dot{\nu}\mu}=U_{\dot{\nu}\mu}[1+K_{\dot{\nu}\mu}I_{\dot{\nu}\mu}]\ , \tag{4.7}$$

which is reduced, in terms of the transposed matrices $\bar{h}_{\dot{\nu}\mu}$ and $\bar{\pi}_{\dot{\nu}\mu}$ of $h_{\dot{\nu}\mu}$ and $\pi_{\dot{\nu}\mu}$, respectively, by

$$\bar{h}_{\dot{\nu}\mu}=U_{\dot{\nu}\mu}K_{\dot{\nu}\mu},\quad \bar{\pi}_{\dot{\nu}\mu}=1-\bar{h}_{\dot{\nu}\mu}\ , \tag{4.8}$$

to the function equations

$$\bar{\pi}_{\dot{\nu}\mu}\check{I}=\check{I}\pi_{\dot{\nu}\mu}=\check{U}_{\dot{\nu}\mu}\ , \tag{4.9}$$

in the same form as Eq. (4.3) for $\check{S}$. The functions $\check{I}$ and $\check{S}$ are connected with each other by relations (3.19), which can be unified to be written by

$$K_{\dot{\nu}\mu}\check{I}=U^{\dot{\nu}\mu}\check{S},\quad K^{\dot{\nu}}_{\mu}\check{I}=U^{\mu}_{\dot{\nu}}\check{S}\ , \tag{4.10}$$

with the superscripts $\dot{\nu}$ and μ, which stand for the subscripts with the complementary odd and even numbers, respectively. Equations (4.6) provide another version of the same relations, yielding

$$h_{\dot{\nu}\mu}\check{S}=h^{\dot{\nu}\mu}\check{S}=K_{\dot{\nu}\mu}K^{\dot{\nu}\mu}\check{I}\ ; \tag{4.11}$$

the counterpart set of equations in which the roles of $\check{I}$ and $\check{S}$ are interchanged are

$$\bar{h}_{\dot{\nu}\mu}\check{I}=\bar{h}^{\dot{\nu}\mu}\check{I}=U^{(4)}\check{S}\ . \tag{4.12}$$

The $\check{K}_{\dot{\nu}\mu}$'s are expressed in terms of $\check{S}$ through Eqs. (4.4), which are expressed by

$$\check{K}_{\dot{\nu}\mu}=\check{K}^0+h^{\mu}_{\dot{\nu}}\check{S} \tag{4.13a}$$

$$=\check{K}^0+K^{\mu}_{\dot{\nu}}K^{\dot{\nu}}_{\mu}\check{I}=\check{K}^{\dot{\nu}\mu}\ . \tag{4.13b}$$

Here the second expression has been written in a symmetrical form, with the aid of (4.11), and the last relation is the reproduction of the conventional reciprocity. $\check{K}^0$ is fully irreducible, subject to the condition (3.16), i.e.,

$$\check{K}^0=\check{K}^0_{\dot{\nu}\mu}\ . \tag{4.14}$$

A manifestly symmetrical expression of $\check{S}$ is obtained by rewriting Eq. (4.5) as

$$\check{S}=\check{K}^0+(h_{12}+h_{14})\check{S}+h_{32}(\check{S}-\check{K}_{14}-h_{14}\check{S}) +h_{34}(\check{S}-\check{K}_{12}-h_{12}\check{S})\ . \tag{4.15}$$

Here the last two terms are identically zero in view of Eq. (4.2). Hence

$$\check{S}=(h_{12}+h_{14}+h_{34}+h_{32})\check{S}-(h_{12}h_{34}+h_{14}h_{32})\check{S}+\check{J}\ , \tag{4.16}$$

where

$$\check{J}=\check{K}^0-K^{(4)}\check{U}\ , \tag{4.17}$$

and works as a source term when solving Eq. (4.16) with, say, the approximations $K_{\dot{\nu}\mu}\approx K^0_{\dot{\nu}\mu}$ and $h_{\dot{\nu}\mu}\approx h^0_{\dot{\nu}\mu}$. The first term on the right-hand side of (4.16) is then linear in $h^0_{\dot{\nu}\mu}$ and the second term works to eliminate the excess terms of doubled diagrams to be made by the first term. Equation (4.16) can be rewritten in terms of $\pi_{\dot{\nu}\mu}$ of (4.1b) as

$$\check{S}=(h_{12}\pi_{34}+h_{34}\pi_{12}+h_{14}\pi_{32}+h_{32}\pi_{14})\check{S} +(h_{12}h_{34}+h_{14}h_{32})\check{S}+\check{J}\ . \tag{4.18}$$

Here we can directly observe that Eq. (4.18) is equivalent to Eqs. (3.24), the sum of the first term and $\check{J}$ being reduced to $K^{(1)}$ of (3.24b) in view of relations (4.3). It may be noticed, however, that the two equations are *not* equivalent to each other when making the approximation $K_{\dot{\nu}\mu}\approx K^0_{\dot{\nu}\mu}$ on the respective right-hand sides, with the expectation that Eq. (4.16) provides an improved version of the original equation (4.5), as long as the same approximation is employed on the both right-hand sides. Each term on the right-hand side of (4.18) has a particular meaning as follows: the term $h_{12}\pi_{34}\check{S}$ is the U_{12}-reducible but U_{34}-irreducible part of $\check{S}$, the term $h_{34}\pi_{12}\check{S}$ is similarly the U_{34}-reducible but U_{12}-irreducible part of $\check{S}$, the term $h_{12}h_{34}\check{S}$ is the reducible part for both U_{12} and U_{34}, and all the other terms are the irreducible parts for the both. The same is also true for another pair of combination U_{14} and U_{32} constituting $U^{(4)}$ of (3.12).

V. STRUCTURE OF K^0

In expression (3.17) for $\check{S}$, the term $\check{K}^0$ is irreducible with respect to U_{12}, U_{34}, U_{14}, and U_{32}, but still reducible with respect to $U_{13}=G^*(1)G^*(3)$ and $U_{24}=G(2)G(4)$ and can be divided into a few reducible and irreducible parts of them by following the procedure to derive the preceding expressions for $\check{S}$. We first write $\check{S}$ by two equations as

$$\check{S}=\check{K}'_{13}+K_{13}U_{13}\check{S} \tag{5.1a}$$

$$=\check{K}'_{24}+K_{24}U_{24}\check{S}\ , \tag{5.1b}$$

with a U_{13}-irreducible matrix K_{13} and a U_{24}-irreducible matrix K_{24} [Fig. 8(a)], in the same fashion as Eqs. (3.11a) and (3.14). Here $K_{13}=K_{31}$ is an $\hat{\mathbf{x}}_1$-$\hat{\mathbf{x}}_3$ symmetrical matrix and $K_{24}=K_{42}$ is likewise an $\hat{\mathbf{x}}_2$-$\hat{\mathbf{x}}_4$ symmetrical matrix. Unlike $\check{K}_{12}$ from K_{12}, the four-coordinate function $\check{K}'_{13}$ is *not* directly connected to the matrix elements of $\check{K}_{13}$ and simply means the U_{13}-irreducible part of $\check{S}$, which is possibly reducible for all the $U_{\dot{\nu}\mu}$'s and U_{24} [see (5.14)]; similarly, $\check{K}'_{24}$ simply means the U_{24}-irreducible part of $\check{S}$. Here, to write $\check{K}^0$, we introduce a symmetrical function

$$\check{K}^{00}\equiv\check{K}^{00}_{12}=\check{K}^{00}_{34}=\check{K}^{00}_{14}=\check{K}^{00}_{32}\ , \tag{5.1c}$$

which is irreducible with respect to U_{13}, U_{24}, and all the $U_{\dot{\nu}\mu}$'s, and express $\check{K}^0$ by

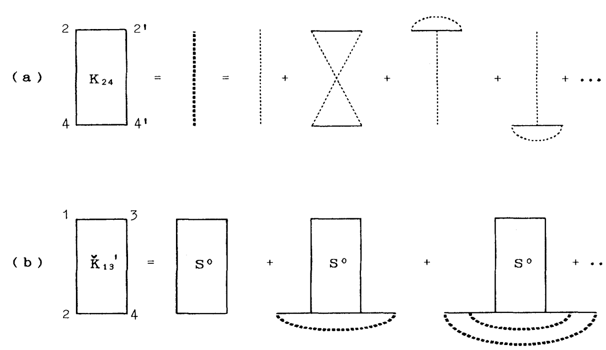

FIG. 8. (a) First four lowest-order diagrams of the matrix K_{24} are shown, which is defined by (5.1b) and represented by a bold broken line in (b). The dotted lines represent K^{00}_{ij}, $i,j=1,2,3,4$. (b) First three diagrams of the series (5.14a) for $\check{K}'_{13}$.

$$\check{K}^0=\check{K}^{00}+K_{24}U_{24}\check{K}'_{13}+K_{13}U_{13}\check{K}'_{24}+K_{13}U_{13}K_{24}U_{24}\check{K}\ , \tag{5.2}$$

in the same fashion as Eq. (4.18) for $\check{S}$; i.e., the second term is U_{24} reducible but U_{13} irreducible, the third term is similarly U_{13} reducible but U_{24} irreducible, the last term is reducible for both U_{13} and U_{24}, and, as it is required, the whole right-hand side is irreducible for all the $U_{\dot{\nu}\mu}$'s for the same reason that the right-hand side of (3.15) is U_{12} irreducible. By using Eqs. (5.1), the factors $\check{K}'_{13}$ and $\check{K}'_{24}$ can be eliminated from (5.2) to obtain a symmetrical expression of $\check{K}^0$, as

$$\check{K}^0=\check{K}^{00}+(h_{13}+h_{24}-h_{13}h_{24})\check{S} \tag{5.3a}$$

$$=\check{K}^{00}+\check{S}-\pi_{13}\pi_{24}\check{S}\ , \tag{5.3b}$$

in terms of the notations

$$h_{ji}=h_{ij}=K_{ij}U_{ij},\quad i,j=1,2,3,4 \tag{5.4a}$$

$$\pi_{ji}=\pi_{ij}=1-h_{ij}=1-K_{ij}U_{ij}\ , \tag{5.4b}$$

which are similar to those by (4.1a) and (4.1b), except here latin subscripts are used to refer to both the odd and the even numbers without distinction. Thus Eq. (4.16) is written, upon substituting the expression (5.3a) for $\check{K}^0$ into $\check{J}$, as

$$\check{S}=(h_{12}+h_{14}+h_{32}+h_{34}+h_{13}+h_{24})\check{S}$$
$$-(h_{12}h_{34}+h_{14}h_{32}+h_{13}h_{24})\check{S}+\check{J}^0\ . \tag{5.5}$$

Here

$$\check{J}^0=\check{K}^{00}-K^{(4)}\check{U}\ , \tag{5.6}$$

which differs from $\check{J}$ only by the term $\check{K}^0$ being changed to $\check{K}^{00}$. In Eq. (5.5), the four coordinates are involved on exactly the same footing. Briefly, the equation can be rewritten as

$$(\pi_{12}\pi_{34}+\pi_{14}\pi_{32}+\pi_{13}\pi_{24}-2)\check{S}=\check{J}^0\ . \tag{5.7a}$$

With the approximations

$$h_{ij}\simeq K^{00}_{ij}U_{ij},\quad \pi_{ij}\simeq 1-K^{00}_{ij}U_{ij}\ , \tag{5.7b}$$

we may regard Eq. (5.5) as a fundamental equation of $\check{S}$ to be solved.

Here it may be worthwhile to confirm that, by setting $K_{13}=K_{24}=0$, Eq. (5.7a) is reduced to the equation

$$(\pi_{12}\pi_{34}+\pi_{14}\pi_{32}-1)\check{S}=\check{J}^0\ , \tag{5.8}$$

which is equivalent to Eq. (4.16), except for the source term $\check{J}$ being changed to $\check{J}^0$; by further setting $K_{14}=K_{32}=0$, it is reduced to

$$\pi_{12}\pi_{34}\check{S}=\check{J}^0=\pi_{34}\check{K}^{00}\ , \tag{5.9}$$

upon using $K_{ij}\approx K^{00}_{ij}$ in $\check{J}^0$. Hence $\pi_{12}\check{S}=\check{K}^{00}_{12}$, which is a reproduction of the original equation (3.11a) to the same approximation on the right-hand side.

To investigate structure of the term $\check{K}'_{13}$, introduced in Eq. (5.1a), in some detail, we first substitute expression (5.3b) of $\check{K}^0$ into the right-hand side of Eq. (4.5) and then write the result in the form

$$\pi_{13}\pi_{24}\check{S}=\check{S}_0 \ . \tag{5.10}$$

Here

$$\check{S}^0=\check{K}^{00}+(h_{12}+h_{14})\check{S} \ , \tag{5.11}$$

which differs from $\check{S}$ of (4.5) only by the first term $\check{K}^0$ being replaced by $\check{K}^{00}$ and is therefore irreducible for both U_{13} and U_{24}. Thus, since, from Eqs. (5.1a) and (5.1b)

$$\check{K}'_{13}=\pi_{13}\check{S}, \quad \check{K}'_{24}=\pi_{24}\check{S} \ , \tag{5.12}$$

we find, using (5.10), that

$$\check{S}=\pi_{13}^{-1}\pi_{24}^{-1}\check{S}^0 \ , \tag{5.13}$$

$$\check{K}'_{13}=\pi_{24}^{-1}\check{S}^0=(1+h_{24}+h_{24}^2+\cdots)\check{S}^0 \ , \tag{5.14a}$$

$$\check{K}'_{24}=\pi_{13}^{-1}\check{S}^0 \ , \tag{5.14b}$$

which show explicitly that K'_{13} contains U_{24}-reducible terms as well as those $U_{\dot{\nu}\mu}$-reducible terms [Fig. 8(b)].

Expression (5.2) for $\check{K}^0$ can be rewritten, in view of Eqs. (5.12), by

$$\check{K}^0=\check{K}^{00}+h_{24}\pi_{13}\check{S}+h_{13}\pi_{24}\check{S}+h_{13}h_{24}\check{S} \ . \tag{5.15}$$

Hence, upon substitution into Eqs. (3.21), we can write a full expression of $\check{K}_{\dot{\nu}\mu}$ in the form

$$\check{K}_{\dot{\nu}\mu}=\check{K}^{00}+\Delta\check{K}_{\dot{\nu}\mu} \ . \tag{5.16a}$$

Here, for example,

$$\Delta\check{K}_{12}=K_{14}K_{32}\check{I}+(h_{24}\pi_{13}+h_{13}\pi_{24}+h_{13}h_{24})\check{S} \ , \tag{5.16b}$$

showing a detailed structure of the $U_{\dot{\nu}\mu}$-irreducible factor $K_{\dot{\nu}\mu}$ to be in the generalized BS equation (4.7). Shown in Fig. 9 is the diagram of $\check{K}^0$ by using Eq. (5.15) with K'_{13} and K'_{24} of Eq. (5.12).

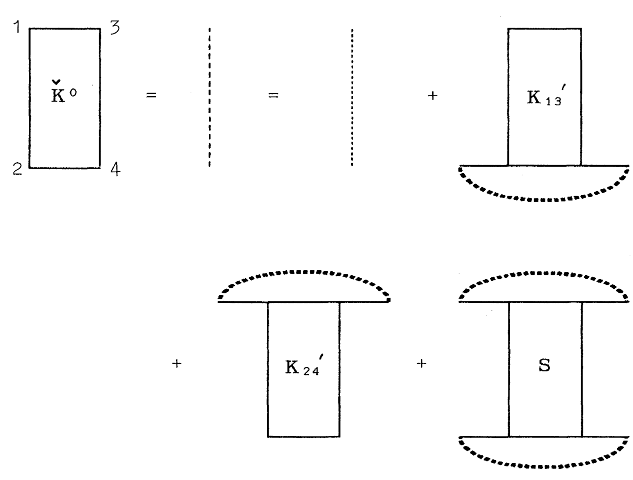

FIG. 9. Structure of $\check{K}^0$ (represented by a broken line in Fig. 4) is shown as a sum of four terms by (5.2). The function $\check{K}^{00}$ is represented by a dotted line.

A. Fine structure of the optical relation

Expression (5.16a) for $K_{\dot{\nu}\mu}$ suggests that the $\hat{x}$ integrated version of optical relation (2.23) can be divided, upon setting $M=M^{00}+\Delta M$ defined by Eq. (2.12) as the whole, as

$$\int d\hat{x}\,\Delta G(\hat{x})K^{00}=(2i)^{-1}[(M^{00})^*-M^{00}] \ , \tag{5.17}$$

$$\int d\hat{x}\,\Delta G(\hat{x})\Delta K=(2i)^{-1}[\Delta M^*-\Delta M] \ , \tag{5.18}$$

in consequence of Eq. (2.25). Equation (5.18) suggests that the enhanced backscattering contributes to the main part of ΔM, in view of $\Delta\check{K}_{12}$ having the structure of (5.16b) whose first term is an exact version of the second term in expression (3.30b) for $\Delta\check{K}=\Delta\check{K}_{12}$ and therefore fully responsible for the enhanced backscattering.

B. Relationship to the Green's function of fourth order

Equation (5.5) for $\check{S}$ is equivalent to the original equation (3.8a) and is nonlinear in view of the coefficients K_{ij} depending on S through Eqs. (4.13) and (5.3). By making the approximation $K_{ij}\approx K_{ij}^{00}$, however, the equation becomes linear and the results can be written in terms of a Green's function I_{1234}, defined as the solution of the integral equation

$$(\bar{\pi}_{12}\bar{\pi}_{34}+\bar{\pi}_{14}\bar{\pi}_{32}+\bar{\pi}_{13}\bar{\pi}_{24}-2)I_{1234}=U^{(4)} \tag{5.19a}$$

or the transposed equation

$$I_{1234}(\pi_{12}\pi_{34}+\pi_{14}\pi_{32}+\pi_{13}\pi_{24}-2)=U^{(4)} \ , \tag{5.19b}$$

with $U^{(4)}$ of (3.12). Here the π_{ij}'s are to be defined by Eq. (5.7b) in terms of K_{ij}^{00}. That is, from Eq. (3.23), the incoherent part $\mathcal{J}$ of I can be written as

$$\check{\mathcal{J}}=U^{(4)}\check{S}=I_{1234}\check{J}^0 \ . \tag{5.20}$$

The proof is given by substituting Eq. (5.7a) for $\check{J}^0$ into the right-hand side of (5.20) and using Eq. (5.19b).

The Green's function I_{1234} can be shown to be the fourth-order Green's function, which is defined by

$$I_{1234}=\langle g^*(1)g(2)g^*(3)g(4)\rangle \tag{5.21}$$

and is governed by Eq. (5.19a), to first order. In the special case in which $K_{32}=K_{14}=K_{13}=K_{24}=0$, Eq. (5.19a) is reduced to $\bar{\pi}_{12}\bar{\pi}_{34}I_{1234}=U^{(4)}$; hence the solution is $I_{1234}=I_{12}I_{34}$, as it should be.

VI. SUMMARY AND DISCUSSION

In principle the boundary condition can always be written in the form (2.4), in terms of the ρ matrix $B^{(12)}$ whose elements are determined once the height change of the boundary surface is given [2]. This enables the original wave equation (2.1) and the boundary equation (2.4) to be unified to be written by one wave equation (2.6) or by its Green's function equation (2.9) written in space-coordinate matrix form, wherein v is an effective medium representing both the medium and the boundary on an equal basis. Thus, with this unified wave equation, basic equations of the statistical Green's functions can be formulated in the same fashion as in an inhomogeneously random medium and in an exact form without depending on any approximation [(2.11)–(2.21) and (2.23)–(2.29)]. The basic matrices M and K are strictly defined by Eqs. (2.12) and (2.19), respectively, and various approximations are possible therefrom. A fixed scatterer q_α embedded in a semi-infinite random layer is considered with the wave equation (3.44). Here the scattering matrix T_α^M of the scatterer q_α in a deterministic medium M, as defined by (3.48), is assumed to be known either theoretically or experimentally, in advance, and it is asked what is its effective change caused by the medium fluctuation, which should lead to the shadowing effect, enhanced backscattering, and other effects resulting from the multiple scattering between the scatterer and the random medium plus the boundary. A specific evaluation of the shadowing and enhanced backscattering effects in this case was made to the diffusion approximation in Ref. [10]. In the case of a semi-infinite random layer, for example, the boundary-value problem of enhanced backscattering is reduced to find the last two terms $K^{(q,12)}$ and $K^{(12,q)}$ in expression (2.20) for K. These terms can be obtained, based on the coordinate-interchange principle, from the incoherent part of the solutions when the normal scattering is assumed throughout and also from the variety of expressions that are available to choose from, as was briefly reviewed with some additional expressions in Sec. II. Here the coordinate interchange procedure is simple when it is made through the optical expressions, with the method as described in Appendix B; the method was previously applied to the case of a random layer to the diffusion approximation [1].

The BS equation of the second-order Green's function can be rewritten as a function equation of the four coordinates involved. The latter formalism enables us to write the basic equations in a manifestly invariant form against arbitrary interchange of the four coordinates $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}_j$, $i \neq j = 1,2,3,4$ (Sec. V), leading to a fundamental function equation written in a fully symmetrical form by Eq. (5.5). On the other hand, making the distinction between the coordinates of the complex conjugate wave functions and those of the original wave functions, the original BS equation (2.18) can be rewritten in a general form by Eq. (4.7), with $K_{\check{\nu}\mu}$ of the form of Eqs. (4.13), wherein the term K^0 is irreducible for all the $U_{\check{\nu}\mu}$'s but is still reducible with respect to U_{13} and U_{24}, having an inner structure as given by (5.3) with the term K^{00}, which is irreducible for all the U_{ij}'s. Thus the irreducible matrix $K_{\check{\nu}\mu}$ is found to have the structure as given by Eqs. (5.16), together with relations (5.12) and (4.3) indicating that the operation of π_{ij} on $\check{S}$ yields the U_{ij}-irreducible part of $\check{S}$. The present formalism naturally leads us to the fourth-order Green's function defined by (5.21) as a basic function, which is a coordinate matrix involving the four coordinates on explicitly the same basis.

APPENDIX A: EXPRESSION OF $T^{(12)}$ IN TERMS OF $M^{(12)}$ AND THE SURFACE GREEN'S FUNCTION

We first introduce the Fourier representation of $G_a^{(0)}(\hat{\mathbf{x}}-\hat{\mathbf{x}}')$ by

$$G_a^{(0)}(\hat{\mathbf{x}}-\hat{\mathbf{x}}')=(2\pi)^{-2}\int d\boldsymbol{\lambda}\exp[-i\boldsymbol{\lambda}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')] \times \tilde{G}_a^{(0)}(z-z') , \quad (A1)$$

with the transform

$$\tilde{G}_a^{(0)}(z-z')=[2i\tilde{h}_a(\boldsymbol{\lambda})]^{-1}\exp[-i\tilde{h}_a(\boldsymbol{\lambda})|z-z'|] . \quad (A2)$$

Here

$$\tilde{h}_a(\boldsymbol{\lambda})=[(k_a^{(M)})^2-\boldsymbol{\lambda}^2]^{1/2} , \quad k_a^{(M)}=(k_a^2+\tilde{M}_a^{(q)})^{1/2}\simeq k_a, \quad \mathrm{Im}[k_a^{(M)}]<0 , \quad (A3)$$

where $\tilde{M}_a^{(q)}(\hat{\boldsymbol{\lambda}})$, $\hat{\boldsymbol{\lambda}}=(\boldsymbol{\lambda},\tilde{h}_a)$, is the Fourier transform of $M_a^{(q)}$ and Im $[\tilde{h}_a]<0$. Hence $G_{ab}(\hat{\mathbf{x}}|\hat{\mathbf{x}}')$ of (2.41) has the Fourier transform $\tilde{G}_{ab}(z|z')$ of the form

$$\tilde{G}_{ab}(z|z')=\tilde{G}_a^{(0)}(z-z')\delta_{ab}+\tilde{G}_a^{(0)}(z)\tilde{T}_{ab}^{(12)}\tilde{G}_b^{(0)}(-z') . \quad (A4)$$

Here

$$\tilde{T}_{ab}^{(12)}=2i\tilde{h}_a\langle R_{ab}^{(12)}\rangle=\tilde{T}_{ba}^{(12)} , \quad (A5)$$

where $\langle R_{ab}^{(12)}\rangle \neq \langle R_{ba}^{(12)}\rangle$ is the reflection-transmission coefficient of the boundary and, when it is perfectly smooth,

$$\langle R_{11}^{(12)}\rangle=\frac{\tilde{h}_1-\tilde{h}_2}{\tilde{h}_1+\tilde{h}_2}, \quad \langle R_{21}^{(12)}\rangle=\frac{2\tilde{h}_1}{\tilde{h}_1+\tilde{h}_2} . \quad (A6)$$

As will be shown shortly, it is generally given in terms of the 2×2 matrix $\tilde{M}^{(12)}$, the Fourier transform of $M^{(12)}$ in the $\boldsymbol{\rho}$ space, by

$$\langle R^{(12)}\rangle=(i\tilde{h}-\tilde{M}^{(12)})^{-1}(i\tilde{h}+\tilde{M}^{(12)}) , \quad (A7)$$

where $\tilde{h}$ is a diagonal matrix with the elements $\tilde{h}_{ab}=\tilde{h}_a\delta_{ab}$. Hence, setting $z=z'=0$ in (A4), the use of Eq. (A5) leads to the surface Green's function

$$\tilde{G}(z=0|z'=0)=(i\tilde{h}-\tilde{M}^{(12)})^{-1} , \quad (A8)$$

in a form similar to the original given by (2.11).

To derive Eq. (A7), we observe that, from Eq. (2.4), the averaged boundary equation is given, in view of the definition for $M^{(12)}$ in (2.13), by

$$-\partial_n^{(a)}\langle\psi_a(\boldsymbol{\rho})\rangle=\sum_{b=1}^{2}\int d\boldsymbol{\rho}' M_{ab}^{(12)}(\boldsymbol{\rho}-\boldsymbol{\rho}')\langle\psi_b(\boldsymbol{\rho}')\rangle \quad (A9)$$

and hence the Fourier transformed version is

$$-\partial_z\langle\tilde{\psi}_1\rangle=\tilde{M}_{11}^{(12)}\langle\tilde{\psi}_1\rangle+\tilde{M}_{12}^{(12)}\langle\tilde{\psi}_2\rangle , \quad +\partial_z\langle\tilde{\psi}_2\rangle=\tilde{M}_{21}^{(12)}\langle\tilde{\psi}_1\rangle+\tilde{M}_{22}^{(12)}\langle\tilde{\psi}_2\rangle . \quad (A10)$$

Here we introduce a particular solution

$$\langle\tilde{\psi}^{(1)}(z)\rangle=\begin{cases}\exp(i\tilde{h}_1 z)+\exp(-i\tilde{h}_1 z)\langle R_{11}\rangle, & z\geq 0\\ \exp[i\tilde{h}_2(z+d_2)]\langle R_{21}\rangle, & z\leq -d_2 ,\end{cases} \quad (A11)$$

written in terms of the reflection-transmission coefficients $\langle R_{11}\rangle$ and $\langle R_{21}\rangle$. Another solution, say, $\langle\tilde{\psi}^{(2)}(z)\rangle$, is obtained by interchanging the role of the subscripts 1 and 2, with the coefficients $\langle R_{22}\rangle$ and $\langle R_{12}\rangle$. Thus those unknown $\langle R_{ij}\rangle$'s can be obtained, on substituting the above two solutions into Eq. (A10), as the solution of a 2×2 matrix equation

$$-i\tilde{h}+i\tilde{h}\langle R\rangle=\tilde{M}^{(12)}(1+\langle R\rangle)\ , \qquad (A12)$$

which leads to Eq. (A7).

APPENDIX B: OPTICAL TRANSFORMATION AND COORDINATE INTERCHANGE

To write equations in optical form, we first introduce relative coordinates $\hat{\mathbf{r}}$ and $\hat{\rho}$, which are defined by

$$\hat{\mathbf{r}}=\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_1,\quad \hat{\rho}=\tfrac{1}{2}(\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_1)\ , \qquad (B1)$$

and the corresponding Fourier variables $\hat{\mathbf{u}}$ and $\hat{\lambda}$, defined by

$$\hat{\mathbf{u}}=\tfrac{1}{2}(\hat{\lambda}_2+\hat{\lambda}_1),\quad \hat{\lambda}=\hat{\lambda}_2-\hat{\lambda}_1\ , \qquad (B2)$$

so that

$$-\hat{\lambda}_1\cdot\hat{\mathbf{x}}_1+\hat{\lambda}_2\cdot\hat{\mathbf{x}}_2=\hat{\mathbf{u}}\cdot\hat{\mathbf{r}}+\hat{\lambda}\cdot\hat{\rho}\ . \qquad (B3)$$

Then we can write the matrix elements of K in the form

$$K(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2')=K(\hat{\mathbf{r}}|\hat{\rho}-\hat{\rho}'|\hat{\mathbf{r}}')\ , \qquad (B4)$$

in view of the translational invariance, approximately in the vertical direction. Here K is usually a short-range function of $\hat{\rho}-\hat{\rho}'$, with a nonzero range of the order of the medium correlation distance. Hence we can write the Fourier transform in the form

$$\tilde{K}(\hat{\lambda}_1;\hat{\lambda}_2|\hat{\lambda}_1';\hat{\lambda}_2')=(2\pi)^3\delta(\hat{\lambda}-\hat{\lambda}')\tilde{K}(\hat{\mathbf{u}}|\hat{\mathbf{u}}')\ , \qquad (B5)$$

the dependence of $\tilde{K}(\hat{\mathbf{u}}|\hat{\mathbf{u}}')$ on $\hat{\lambda}$ being suppressed. The corresponding Fourier transform of $S(\hat{\mathbf{r}},\hat{\rho}|\hat{\mathbf{r}}',\hat{\rho}')$ is written in the original form by $\tilde{S}(\hat{\mathbf{u}},\hat{\lambda}|\hat{\mathbf{u}}',\hat{\lambda}')$. Here, by changing the variable $\hat{\mathbf{u}}$ by $\hat{\mathbf{u}}=u\hat{\Omega}$, $d\hat{\mathbf{u}}=u^2du\,d\hat{\Omega}$, where $u=|\hat{\mathbf{u}}|$ and $\hat{\Omega}=(\Omega,\Omega_z)$, $\hat{\Omega}^2=1$, is the unit vector, the optical expression $S(\hat{\Omega},\hat{\rho}|\hat{\Omega}',\hat{\rho}')$ is obtained therefrom by the Fourier inversion only with respect to $\hat{\lambda}$ and $\hat{\lambda}'$.

As for the Fourier transform $\tilde{U}$ of U, we utilize the relation

$$U(1;2)=[G^*(1)-G(2)]\{G^{-1}(2)-[G^*(1)]^{-1}\} \qquad (B6)$$

to find the expression

$$\tilde{U}(\hat{\mathbf{u}},\hat{\lambda})=\tilde{G}^*(\hat{\mathbf{u}}-\hat{\lambda}/2)\tilde{G}(\hat{\mathbf{u}}+\hat{\lambda}/2)$$
$$\simeq\pi\delta(\hat{\mathbf{u}}^2-k^2)(k\gamma-i\hat{\mathbf{u}}\cdot\hat{\lambda})^{-1}\ . \qquad (B7)$$

Here $k\gg\gamma\gtrsim|\hat{\lambda}|$ and

$$\gamma=(2ik)^{-1}(\tilde{M}^*-\tilde{M})(\hat{\mathbf{u}})\ . \qquad (B8)$$

Hence we obtain an important relation that, for any slowly changing function $f(\hat{\mathbf{u}})$,

$$(2\pi)^{-3}\int d\hat{\mathbf{u}}\,\tilde{U}(\hat{\mathbf{u}},\hat{\lambda})f(\hat{\mathbf{u}})\simeq\int_{4\pi}d\hat{\Omega}\,\tilde{U}(\hat{\Omega},\hat{\lambda})f(\hat{\Omega})\ , \qquad (B9)$$

where

$$\tilde{U}(\hat{\Omega},\hat{\lambda})=(\gamma-i\hat{\Omega}\cdot\hat{\lambda})^{-1}=\tilde{U}(-\hat{\Omega},-\hat{\lambda})\ , \qquad (B10)$$

$$f(\hat{\Omega})=(4\pi)^{-2}f(\hat{\mathbf{u}}=k\hat{\Omega})\ . \qquad (B11)$$

Here the λ_z Fourier inversion of $\tilde{U}(\hat{\Omega},\hat{\lambda})$ is

$$U(\hat{\Omega},z)\equiv(2\pi)^{-1}\int d\lambda_z\exp(-i\lambda_z z)\tilde{U}(\hat{\Omega},\hat{\lambda})$$
$$=\begin{cases}|\Omega_z|^{-1}\exp[-\Omega_z^{-1}(\gamma-i\Omega\cdot\lambda)z], & \Omega_z z>0\\ 0, & \Omega_z z<0\ ,\end{cases} \qquad (B12)$$

while the three-dimensional inversion $U(\hat{\Omega},\hat{\rho})$ is given by

$$U(\hat{\Omega},\hat{\rho})=|\hat{\rho}|^{-2}\exp(-\gamma|\hat{\rho}|)\delta^2(\hat{\Omega}-\hat{\rho}/|\hat{\rho}|)\ ,$$
$$\int d\hat{\Omega}\,\delta^2(\hat{\Omega}-\hat{\Omega}')=1\ , \qquad (B13)$$

where $\delta^2(\hat{\Omega})$ is a two-dimensional δ function with respect to the solid angle $\hat{\Omega}$. Hence, for example, the optical expression of $\tilde{K}\tilde{U}\tilde{S}$ becomes, according to formula (B9)

$$\int d\hat{\Omega}''K(\hat{\Omega}|\hat{\Omega}'')\tilde{U}(\hat{\Omega}'',\hat{\lambda})\tilde{S}(\hat{\Omega}'',\hat{\lambda}|\hat{\Omega}',\hat{\lambda}')\ , \qquad (B14)$$

where

$$\tilde{S}(\hat{\Omega},\hat{\lambda}|\hat{\Omega}',\hat{\lambda}')=(4\pi)^{-2}\tilde{S}(\hat{\mathbf{u}}=k\hat{\Omega},\hat{\lambda}|\hat{\mathbf{u}}'=k\hat{\Omega}',\hat{\lambda}')\ , \qquad (B15)$$

and a similar expression for $K(\hat{\Omega}|\hat{\Omega}')$ by (B11).

The optical expression of $\mathcal{T}_{22}^{(q/12+23)}$ in (3.42), say, $\mathcal{T}_{22}^{(q/12+23)}(\hat{\Omega},\rho|\hat{\Omega}',\rho')$, $z=z'=0$, is obtained by using the expression

$$\mathcal{T}_{22}^{(q/12+23)}=U_2S_{22}^{(q/12+23)}U_2 \qquad (B16)$$

similar to (2.54) and considering its Fourier transform with respect to the ρ coordinates, say, $\tilde{\mathcal{T}}_{22}^{(q/12+23)}(\hat{\Omega}|\lambda|\hat{\Omega}')$, which is given with the aid of (B12) by

$$\tilde{\mathcal{T}}_{22}^{(q/12+23)}(\hat{\Omega}|\lambda|\hat{\Omega}')=\int_{-L}^{0}dz\int_{-L}^{0}dz'\,U_2(\hat{\Omega},-z)\tilde{S}_{22}^{(q/12+23)}(\hat{\Omega},z|\lambda|\hat{\Omega}',z')U_2(\hat{\Omega}',z') \qquad (B17)$$
$$=|\Omega_z\Omega_z'|^{-1}\int_{-L}^{0}dz\int_{-L}^{0}dz'\exp[i\lambda_z z-i\lambda_z'z']\tilde{S}_{22}^{(q/12+23)}(\hat{\Omega},z|\lambda|\hat{\Omega}',z')\ , \qquad (B18)$$

where

$$\lambda_z=-i(\gamma_2-i\Omega\cdot\lambda)\Omega_z^{-1},\quad \Omega_z>0$$
$$\lambda_z'=-i(\gamma_2-i\Omega'\cdot\lambda)(\Omega_z')^{-1},\quad \Omega_z'<0\ . \qquad (B19)$$

To make the coordinate interchange $\hat{\mathbf{x}}_2\leftrightarrow\hat{\mathbf{x}}_4$, we observe from (B3) that the Fourier transformation function is presently

$$\exp[i(-\hat{\lambda}_1\cdot\hat{\mathbf{x}}_1+\hat{\lambda}_2\cdot\hat{\mathbf{x}}_2+\hat{\lambda}_3\cdot\hat{\mathbf{x}}_3-\hat{\lambda}_4\cdot\hat{\mathbf{x}}_4)] \qquad (B20)$$

and the integral (B17), rewritten by

$$\mathcal{J}_{22}^{(q/12+23)}(\hat{\mathbf{\Omega}}_{12}|\boldsymbol{\lambda}_{12}=\boldsymbol{\lambda}_{34}|\hat{\mathbf{\Omega}}_{34}) ,$$

is changed to

$$\mathcal{J}_{22}^{(q/12+23)}(\hat{\mathbf{\Omega}}_{14}|\boldsymbol{\lambda}_{14}=\boldsymbol{\lambda}_{32}|\hat{\mathbf{\Omega}}_{32}) . \tag{B21}$$

Here, from (B2),

$$\begin{aligned} k\hat{\mathbf{\Omega}}_{12}&=\hat{\mathbf{u}}_{12}=\tfrac{1}{2}(\hat{\boldsymbol{\lambda}}_1+\hat{\boldsymbol{\lambda}}_2) ,\\ k\hat{\mathbf{\Omega}}_{34}&=\hat{\mathbf{u}}_{34}=\tfrac{1}{2}(\hat{\boldsymbol{\lambda}}_3+\hat{\boldsymbol{\lambda}}_4) ,\\ k\hat{\mathbf{\Omega}}_{14}&=\hat{\mathbf{u}}_{14}=\tfrac{1}{2}(\hat{\boldsymbol{\lambda}}_1-\hat{\boldsymbol{\lambda}}_4) ,\\ k\hat{\mathbf{\Omega}}_{32}&=\hat{\mathbf{u}}_{32}=\tfrac{1}{2}(\hat{\boldsymbol{\lambda}}_3-\hat{\boldsymbol{\lambda}}_2) , \end{aligned} \tag{B22}$$

and similarly

$$\begin{aligned} \boldsymbol{\lambda}_{12}&=-\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2, \quad \boldsymbol{\lambda}_{34}=-\boldsymbol{\lambda}_3+\boldsymbol{\lambda}_4 ,\\ \boldsymbol{\lambda}_{14}&=-\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_4, \quad \boldsymbol{\lambda}_{32}=-\boldsymbol{\lambda}_3-\boldsymbol{\lambda}_2 , \end{aligned} \tag{B23}$$

with the relation

$$\boldsymbol{\lambda}_{12}-\boldsymbol{\lambda}_{34}=\boldsymbol{\lambda}_{14}-\boldsymbol{\lambda}_{32} , \tag{B24}$$

while the corresponding λ_z's are to be given according to (B19) in terms of the $\boldsymbol{\lambda}$'s. Equations (B22) and (B23) are expressed in terms of the original variables $\hat{\mathbf{\Omega}}_{12}$, $\hat{\mathbf{\Omega}}_{34}$, $\boldsymbol{\lambda}_{12}$, and $\boldsymbol{\lambda}_{34}$ by

$$\begin{aligned} \hat{\mathbf{\Omega}}_{14}&=\tfrac{1}{2}(\hat{\mathbf{\Omega}}_{12}-\hat{\mathbf{\Omega}}_{34})-(4k)^{-1}(\hat{\boldsymbol{\lambda}}_{12}+\hat{\boldsymbol{\lambda}}_{34}) ,\\ \hat{\mathbf{\Omega}}_{32}&=\tfrac{1}{2}(\hat{\mathbf{\Omega}}_{34}-\hat{\mathbf{\Omega}}_{12})-(4k)^{-1}(\hat{\boldsymbol{\lambda}}_{12}+\hat{\boldsymbol{\lambda}}_{34}) , \end{aligned} \tag{B25}$$

$$\begin{aligned} \boldsymbol{\lambda}_{14}&=-k(\mathbf{\Omega}_{12}+\mathbf{\Omega}_{34})+\tfrac{1}{2}(\boldsymbol{\lambda}_{12}-\boldsymbol{\lambda}_{34}) ,\\ \boldsymbol{\lambda}_{32}&=-k(\mathbf{\Omega}_{12}+\mathbf{\Omega}_{34})+\tfrac{1}{2}(\boldsymbol{\lambda}_{34}-\boldsymbol{\lambda}_{12}) . \end{aligned} \tag{B26}$$

In the case of a fixed scatterer (Sec. III), the translational invariance does not hold any longer and $\boldsymbol{\lambda}_{12}$ and $\boldsymbol{\lambda}_{34}$ work as independent Fourier variables of integration, in contrast to the transform (B21). The integral was specifically evaluated to the diffusion approximation in Ref. [10].

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